MA 101 (Mathematics-I)

Multivariable Calculus part 2: Practice Problem Sheet 2

1. Consider the transformation $T : [0, 2\pi]\times[0, 1] \to \mathbb{R}^2$ given by $T(u, v) = (2v \cos u, v \sin u)$.
   (a) For a fixed $v_o \in [0, 1]$, describe the set $\{T(u, v_o) : \ u \in [0, 2\pi]\}$.
   (b) Describe the set $\{T(u, v) : \ [0, 2\pi] \times [0, 1]\}$.

2. Let $R$ be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x,\ y = 3x$ and $x + y = 4$. Consider the transformation $T(u, v) = (u-v, u+v)$. Find the set $S$ satisfying $T(S) = R$.

3. Evaluate $\iint\limits_R x dx dy$ where $R$ is the region $1 \le x(1 - y) \le 2$ and $1 \le xy \le 2$.

4. Evaluate
   (a) $\int\limits_0^{\frac{1}{\sqrt{2}}} \int\limits_{x=y}^{\sqrt{1-y^2}} (x + y) dx dy.$
   (b) $\int\limits_0^{1} \int\limits_{x=0}^{1-y} \sqrt{x + y}(y - 2x)^2 dx dy.$
   (c) $\int\limits_1^{2} \int\limits_{x=0}^{y} \frac{1}{(x^2+y^2)^{\frac{3}{2}}} dx dy.$
   (d) $\int\limits_0^{2} \int\limits_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy dx.$

5. Using change of variables $u = x + y$ and $v = x - y$, show that

$$\int\limits_0^1 \int\limits_{y=0}^{y=x} (x - y) dy dx = \int\limits_0^1 \int\limits_{u=v}^{2-v} \frac{v}{2} du dv.$$

6. Find the volume of the solid in the first octant bounded below by the surface $z = \sqrt{x^2 + y^2}$ above by $x^2 + y^2 + z^2 = 8$ as well as the planes $y = 0$ and $y = x$.

7. Find the volume of the solid bounded by the surfaces $z = 3(x^2+y^2)$ and $z = 4-(x^2+y^2)$.

8. Let $D$ denote the solid bounded by surfaces $y = x,\ y = x^2,\ z = x$ and $z = 0$. Evaluate $\iiint\limits_D y dx dy dz$.

9. Let $D$ denote the solid bounded below by the plane $z + y = 2$, above by the cylinder $z + y^2 = 4$ and on the sides $x = 0$ and $x = 2$. Evaluate $\iiint\limits_D x dx dy dz$.

10. Let $D = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \le 1\}$ and $E = \{(u, v, w) \in \mathbb{R}^3 : u^2+v^2+w^2 \le 1\}$. Show that $\iiint\limits_D dx dy dz = \iiint\limits_E 24 du dv dw$.

11. Let $D$ be the solid that lies inside the cylinder $x^2 + y^2 = 1$, below the cone $z = \sqrt{4(x^2 + y^2)}$ and above the plane $z = 0$. Evaluate $\iiint\limits_D x^2 dxdydz$.

12. Evaluate $\int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{x^2+y^2}^{4} xdzdydx$.

13. Let $D$ denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2 + y^2 + z^2} dxdydz$.

14. Parametrize the part of the sphere $x^2 + y^2 + z^2 = 16$, $-2 \le z \le 2$ using the spherical co-ordinates.

15. Let $D$ denote the solid enclosed by the spheres $x^2+y^2+(z-1)^2 = 1$ and $x^2+y^2+z^2 = 3$. Using the spherical coordinates, set up iterated integral that gives the volume of $D$.

16. Let $S$ be the part of the sphere $x^2+y^2+z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Parametrize $S$ by considering it as a graph and again by using the spherical coordinates.

17. Let $S$ denote the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. Find the area of $S$.

    (a) By considering $S$ as a part of the graph $z = f(x, y)$, where $f(x, y) = 10 - 2x - 5y$.

    (b) By considering $S$ as a parametric surface $r(u, v) = (u \cos v, u \sin v, 10 - u(2 \cos v + 5 \sin v))$, $0 \le u \le 3$ and $0 \le v \le 2\pi$.

18. Find the area of the surface $x = uv, y = u + v, z = u - v$, where $(u, v) \in D = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 \le 1\}$.

19. Find the area of the part of the surface $z = x^2 + y^2$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.

20. Let $S$ be the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant. Evaluate $\iint\limits_S (z + 2xy)d\sigma$.