MA 101 (Mathematics-I)

Multivariable Calculus part 2: Practice Problem Sheet 3

1. Let $\vec{N}$ be the unit outward normal vector on the ellipse $x^2 + 2y^2 = 1$. Evaluate the line integral $\int\limits_C \vec{N}.d\vec{R}$ along the unit circle $C = \{(x, y) : \ x^2 + y^2 = 1\}$.

2. Use fundamental theorem of calculus for line integral to show that $\int_C y\,dx + (x+z)\,dy + y\,dz$ is independent of any path C joining the points $(2, 1, 4)$ and $(8, 3, -1)$.

3. Consider the curve C which is the intersection of the surfaces $x^2 + y^2 = 1$ and $z = x^2$. Assume that C is oriented counterclockwise as seen from the positive z-axis. Evaluate $\int\limits_C zdx - xydy - xdz$.

4. Let $f(x, y, z) = (x^2, xy, 1)$. Show that that there is no ϕ such that $\nabla\phi = f$.

5. Let C be a curve represented by two parametric representations such that $C = \{R_1(s) : \ s \in [a, b]\} = \{R_2(t) : \ t \in [c, d]\}$, where $R_1 : [a, b] \to \mathbb{R}^3$ and $R_2 : [c, d] \to \mathbb{R}^3$ be two distinct differentiable one-one maps.

   (a) Show that there exists a function $h : [c, d] \to [a, b]$ such that $R_2(t) = R_1(h(t))$.

   (b) If R_1 and R_2 trace out C in the same direction, then $\int\limits_C f.dR_1 = \int\limits_C f.dR_2$.

   (c) If R_1 and R_2 trace out C in the opposite direction, then $\int\limits_C f.dR_1 = -\int\limits_C f.dR_2$.

6. Evaluate the line integral $\oint\limits_C (x^2 \sin^2 x - y^3)dx + (y^2 \cos^2 y - y)dy$, where C is the closed curve consisting $x + y = 0$, $x^2 + y^2 = 25$ and $y = x$ and lying in the first and fourth quadrants.

7. Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuously differentiable function. Suppose C is the boundary of the region bounded above by the graph of f, below by the x-axis and on the sides by the lines $x = a$ and $x = b$. Show that $\int\limits_a^b f(x)dx = -\oint\limits_C ydx$.

8. Let $F(x, y, z) = (y, -x, 2z^2 + x^2)$ and S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$. Evaluate $\iint\limits_S \operatorname{curl} F \cdot \hat{n} d\sigma$, where $\hat{n}$ is the unit outward normal of S.

9. Let C be the boundary of the cone $z = x^2 + y^2$ and $0 \leq z \leq 1$. Use Stoke's theorem to evaluate the line integral $\int_C \vec{F}.d\vec{R}$ where $\vec{F} = (y, xz, 1)$.

10. Let $\vec{F} = (xy, yz, zx)$ and S be the surface $z = 4 - x^2 - y^2$ with $2 \leq z \leq 4$. Use divergence theorem to find the surface integral $\iint\limits_S \vec{F}.\vec{n} dS$.

11. Let S be the sphere $x^2 + y^2 + z^2 = 1$. If some $\alpha \in \mathbb{R}$ satisfies $\iint\limits_S (zx + \alpha y^2 + xz) d\sigma = \frac{4\pi}{3}$, then find α.