Elementary properties of Complex numbers

Introduction

- Let us consider the quadratic equation $x^2 + 1 = 0$.
- It has no real root.
- Let i(iota) be the solution of the above equation, then
  - $i^2 = -1$ i.e. $i = \sqrt{-1}$.
  - i is not a real number. So we define it as *imaginary number*.
- A complex number is defined by $z = x + iy$, for any $x, y \in \mathbb{R}$.
- Complex analysis is theory of functions of complex numbers.

Application of Complex Analysis

- Why do we need Complex Analysis?
- Evaluation of certain integrals which are difficult to workout. Viz.

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

- Fourier Analysis.
- Differential Equations.
- Number Theory.
- All major branches of Mathematics which is applicable in science and engineering.

Complex Numbers

- A complex number denoted by z is an ordered pair (x, y) with $x \in \mathbb{R}$, $y \in \mathbb{R}$.
- x is called real part of z and y is called the imaginary part of z. In symbol $x = \text{Re } z$, and $y = \text{Im } z$.
- We denote $i = (0, 1)$ and hence we write $z = x + iy$ where the element x is identified with $(x, 0)$, and y is identified with $(0, y)$.
- $\text{Re } z = \text{Im } iz$ and $\text{Im } z = -\text{Re } iz$.
- By $\mathbb{C}$ we denote the set of all complex numbers, that is, $\mathbb{C} = \{z : z = x + iy, x \in \mathbb{R}, y \in \mathbb{R}\}$.

Algebra of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

- **Addition and subtraction**: We define

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2).$$

- **Multiplication**: We define

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

  Since $i = (0, 1)$ it follows from above that $i^2 = -1$.

- **Division**: If z a non-zero complex number then we define

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

  From this we get

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{{x_2}^2 + {y_2}^2}.$$

Basic algebraic properties of Complex Numbers

Let $z_1, z_2, z_3 \in \mathbb{C}$.

- **Commutative and associative law for addition** : $z_1 + z_2 = z_2 + z_1$. and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
- **Additive identity** : $z + 0 = 0 + z = z \;\forall\; z \in \mathbb{C}$
- **Additive inverse** : For every $z \in \mathbb{C}$ there exists $-z \in \mathbb{C}$ such that $z + (-z) = 0 = (-z) + z$.
- **Commutative and associative law for multiplication** : $z_1 z_2 = z_2 z_1$. and $z_1(z_2 z_3) = (z_1 z_2) z_3$.
- **Multiplicative identity** : $z \cdot 1 = z = 1 \cdot z \;\forall\; z \in \mathbb{C}$
- **Multiplicative inverse** : For every non-zero $z \in \mathbb{C}$ there exists $w(= \frac{1}{z}) \in \mathbb{C}$ such that $zw = 1 = wz$.
- **Distributive law** : $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Note: $\mathbb{C}$ **is a field.**

Conjugate of a Complex Number

If $z = x + iy$ is a complex number then its **conjugate** is defined by $\bar{z} = x - iy$. Conjugation has the following properties which follows easily from the definition. Let $z_1, z_2 \in \mathbb{C}$ then,

- Re $z = \frac{1}{2}(z + \bar{z})$ and Im $z = \frac{1}{2i}(z - \bar{z})$.
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- Note: If $\alpha \in \mathbb{R}$ then $\overline{\alpha z} = \alpha \bar{z}$).
- $\bar{\bar{z}} = z$
- Re $z =$ Re $\bar{z}$ and Im $z = -$Im $\bar{z}$.

Modulus of a Complex Number

The **modulus** or **absolute** value of a complex number $z = x + iy$ is a non-negative real number denoted by $|z|$ and defined by

$$|z| = \sqrt{x^2 + y^2}.$$

Note that if $z = x + iy$ then $|z|$ is the Euclidean distance of the point (x, y) from the origin $(0, 0)$.

Exercise: Verify the following properties.

- $z\bar{z} = |z|^2$.
- $|x| = |\text{Re } z| \leq |z|$ and $|y| = |\text{Im } z| \leq |z|$
- $|\bar{z}| = |z|, |z_1 z_2| = |z_1||z_2|$ and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}(z_2 \neq 0)$.
- $|z_1 + z_2| \leq |z_1| + |z_2|$(**Triangle inequality**).
- $||z_1| - |z_2|| \leq |z_1 - z_2|$

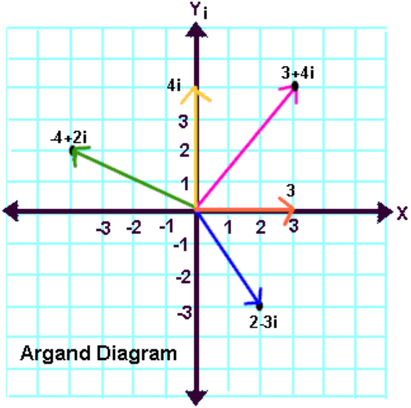
Graphical representation of Complex Numbers

- We can represent the complex number $z = x + iy$ by a position vector in the $XY-$plane whose tail is at the origin and head is at the point (x, y).
- When $XY-$plane is used for displaying complex numbers, it is called **Argand plane** or **Complex plane** or **z plane**.
- The X-axis is called as the real axis where as the Y-axis is called as the imaginary axis.

Graph the complex numbers:

1. **$3 + 4i$** (3,4)
2. **$2 - 3i$** (2,-3)
3. **$-4 + 2i$** (-4,2)
4. **3** (which is really $3 + 0i$) (3,0)
5. **4i** (which is really $0 + 4i$) (0,4)

The complex number is represented by the point or by the vector from the origin to the point.



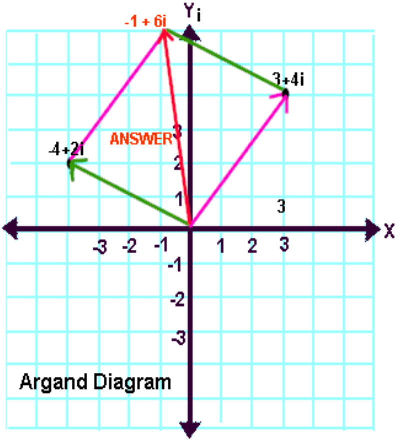
Add $3 + 4i$ and $-4 + 2i$ graphically.

Graph the two complex numbers $3 + 4i$ and $-4 + 2i$ as vectors.

Create a parallelogram using these two vectors as adjacent sides.

The sum of $3 + 4i$ and $-4 + 2i$ is represented by the diagonal of the parallelogram (read from the origin).

This new (diagonal) vector is called the resultant vector.



Graphical representation of Complex Numbers

Subtract 3 + 4*i* from -2 + 2*i*

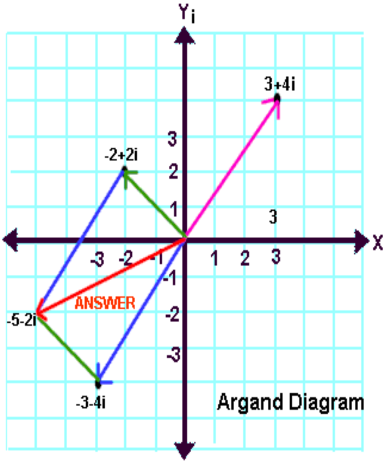
Subtraction is the process of adding the additive inverse.

$(-2 + 2i) - (3 + 4i)$
$= (-2 + 2i) + (-3 - 4i)$
$= (-5 - 2i)$

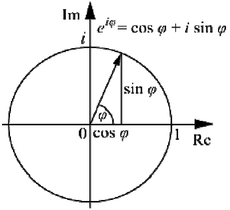
Graph the two complex numbers as vectors.

Graph the additive inverse of the number being subtracted.

Create a parallelogram using the first number and the additive inverse. The answer is the vector forming the diagonal of the parallelogram.



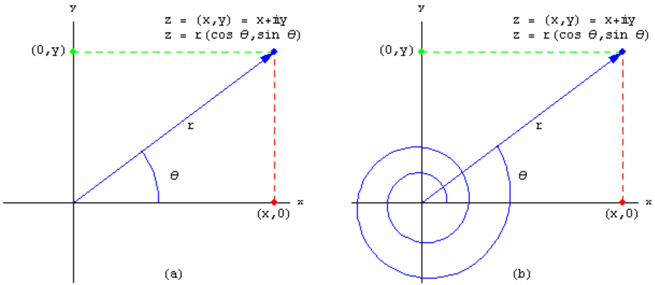
Polar representation of Complex Numbers



- Consider the unit circle on the complex plane. Any point on the unit circle is represented by $(\cos\varphi, \sin\varphi), \varphi \in [0, 2\pi]$.
- Any non-zero $z \in \mathbb{C}$, the point $\dfrac{z}{|z|}$ lies on the unit circle and therefore we write $\dfrac{z}{|z|} = \cos\varphi + i\sin\varphi$. i.e. $z = |z|(\cos\varphi + i\sin\varphi)$.
- The symbol $e^{i\varphi}$ is defined by means of *Euler's formula* as

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

Polar representation of Complex Numbers



- Any non-zero $z = x + iy$ can be uniquely specified by its magnitude(length from origin) and direction(the angle it makes with positive $X-$axis).
- Let $r = |z| = \sqrt{x^2 + y^2}$ and θ be the angle made by the line from origin to the point (x, y) with the positive $X-$axis.
- From the above figure $x = r\cos\theta,\ y = r\sin\theta$ and $\theta = \tan^{-1}(\frac{y}{x})$.

Polar representation of a Complex Number

- If $z \neq 0$ then $\arg(z) = \{\theta : z = re^{i\theta}\}$.
- Note that $\arg(z)$ is a multi-valued function.

$$\arg(z) = \{\theta + 2n\pi : z = re^{i\theta}, n \in \mathbb{Z}\}.$$

- $\arg z = \text{Arg } z + 2k\pi$ So, if θ is argument of z then so is $\theta + 2k\pi$. For example, $\arg i = 2k\pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$, where as $\text{Arg } i = \frac{\pi}{2}$.
- The principal value of $\arg(z)$, denoted by $\text{Arg}(z)$, is the particular value of $\arg(z)$ chosen in within $(-\pi, \pi]$.
- Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$ then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.
- If $z_1 \neq 0$ and $z_2 \neq 0$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
- As $|e^{i\theta}| = 1, \ \forall \ \theta \in \mathbb{R}$, it follows that $|z_1 z_2| = |z_1||z_2|$.

- **De Moiver's formula:**

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

- **Problem:** Given a non-zero complex number z_0 and a natural number $n \in \mathbb{N}$. Find all distinct complex numbers w such that $z_0 = w^n$.
- If w satisfies the above then $|w| = |z_0|^{\frac{1}{n}}$. So, if $z_0 = |z_0|(\cos\theta + i\sin\theta)$ we try to find α such that

$$|z_0|(\cos\theta + i\sin\theta) = [|z_0|^{\frac{1}{n}}(\cos\alpha + i\sin\alpha)]^n.$$

- By De Moiver's formula $\cos\theta = \cos n\alpha$ and $\sin\theta = \sin n\alpha$, that is, $n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}$. The distinct values of w is given by $|z_0|^{\frac{1}{n}}(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n})$, for, $k = 0, 1, 2, \ldots, n-1$.