Topology of Complex plane (Open and Closed set)

Some Basic Definitions

- **Open Disc**: Let $z_0 \in \mathbb{C}$ and $r > 0$ then, $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is called an open disc centered at z_0 with radius r.
- **Deleted Neighborhood of z_0 :** Let $z_0 \in \mathbb{C}$ and $r > 0$ then, $B(z_0, r) - \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ is called the **deleted neighborhood** of z_0.
- **Interior point:** A point z_0 is called an **interior point** of a set $S \subset \mathbb{C}$ if we can find an $r > 0$ such that $B(z_0, r) \subset S$. The set of all interior points of S is denoted by S°.
- **Boundary points:** If $B(z_0, r)$ contains points of S and points of S^c every $r > 0$, then z_0 is called a **boundary point** of a set S. The set of all boundary points of S is denoted by ∂S.
- **Exterior points:** If a point is not an interior point or boundary point of S, it is an exterior point of S. The set of all exterior points of S is denoted by $\text{ext}(S)$.

Some Basic Definitions

- **Open Set:** A set $S \subset \mathbb{C}$ is **open** if every $z_0 \in S$ there exists $r > 0$ such that $B(z_0, r) \subset S$.
- **Exercise:** Show that a set S is an open set if and only if every point of S is an interior point.
- **Connected Set*:** An open set $S \subset \mathbb{C}$ is said to be **connected** if each pair of points z_1 and z_2 in S can be joined by a polygonal line consisting of a finite number of line segments joined end to end that lies entirely in S.
- * Please refer to the last page for standard definition of connectedness. However, as far as this course is concerned, we will stick to the above definition of connectedness.
- **Domain/Region:** An open, connected set is called a **domain**. A domain together with some, none or all of its boundary points is called **region**.

Some Basic Definitions

- **Bounded Set:** A set $S \subset \mathbb{C}$ is **bounded** if there exists a $K > 0$ such that $|z| < K \ \forall \ z \in S$. We say S is **unbounded** if S is not bounded.
- **Limit point/Accumulation point:** A point $\zeta \in \mathbb{C}$ is called a **limit point** of a set $S \subset \mathbb{C}$ if every deleted neighborhood of ζ contains at least one point of S. The set of all limit points of S is denoted by S'.
- **Closed Set:** A set $S \subset \mathbb{C}$ is said to be **closed** if S contains all its limit points.
- **Exercise:** Show that a set S is closed if and only if S^c is open.
- **Closure of a set:** The **closure** of a set $S \subset \mathbb{C}$, denoted by $\bar{S}$, is defined as the set S together with all its limit points.
- **Exercise:** Show that a set S is closed if and only if $\bar{S} = S$.
- **Exercise:** Show that the exterior of a subset S of $\mathbb{C}$ is the complement of the closure of the set S.

Standard definition of connected sets

- **Path:** A path is continuous function $\gamma : [a, b] \to \mathbb{C}$. Every polygonal path is also a path.
- **Connected set:** A subset S of $\mathbb{C}$ is said to be disconnected if there are disjoint open sets A and B such that $A \cup B = S$. A set is connected if it is not disconnected.
- **Result:** Let S be a connected subset of $\mathbb{C}$, and $B = S \cup \{$**some** *or* **all** limits points of S$\}$. Then B is connected.
- **Path connected set:** A subset S of $\mathbb{C}$ is said to be path connected if each pair of points z_1 and z_2 in S can be joined by a path lying entirely in S.
- **Result:** Path connectedness $\Longrightarrow$ connectedness but **connectedness** $\not\Longrightarrow$ **path connectedness**.
- **Example: (Comb space)** Let $\mathcal{C} = ([0,1] \times 0) \bigcup \left(\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n} \times [0,1] \right) \right) \cup \{(0,1)\}$. Then $\mathcal{C}$ is connected but not path connected.
- **Result:** If S in $\mathbb{C}$ is open, then path connectedness $\Longleftrightarrow$ connectedness.