Sequence, Limit and Continuity

Functions of a complex variable

- Let $S \subseteq \mathbb{C}$. A **complex valued function** is a rule that assigns to each complex number $z \in S$ a unique complex number w.
- We write $w = f(z)$. The set S is called the **domain of** f and the set $\{f(z) : z \in S\}$ is called **range** of f.
- Any complex function can be separated into real and imaginary parts:

$$z = x + iy \quad \text{and} \quad w = f(z) = u(x, y) + iv(x, y),$$

where $x, y \in \mathbb{R}$ and $u(x, y), v(x, y)$ are **real-valued** functions. In other words, the components of the function $f(z)$, $u = u(x, y)$ and $v = v(x, y)$ can be interpreted as **real-valued functions** of the two real variables x and y.

Complex Sequences

- **Complex Sequences:** A **complex sequence** is a function whose domain is the set of natural numbers and range is a subset of complex numbers. **In other words, a sequence can be written as $f(1), f(2), f(3) \ldots$. Usually, we will denote such a sequence by the symbol $\{z_n\}$, where $z_n = f(n)$.**
- **Limit of a sequence:** A number l is called the **limit** of an infinite sequence $\{z_n\}$, if for every $\epsilon > 0$, there exists a $N_\epsilon > 0$ such that $|z_n - l| < \epsilon$ whenever $n \geq N_\epsilon$.
- In such case we write $\lim_{n\to\infty} z_n = l$.
- If the limit of the sequence exists we say that the sequence is **convergent**; otherwise it is called not convergent.
- A convergent sequence has a **unique** limit.
- Every convergent sequence is **bounded**.
- If $z_n = x_n + iy_n$ and $l = \alpha + i\beta$ then

$$\lim_{n\to\infty} z_n = l \iff \lim_{n\to\infty} x_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} y_n = \beta.$$

Limit of a function

- **Limit of a function:** Let f be a complex valued function defined at all points z in some deleted neighborhood of z_0. We say that f has a **limit** a as $z \to z_0$ and write
$$\lim_{z \to z_0} f(z) = a,$$
if for every $\epsilon > 0$, there is a $\delta > 0$ such that
$$|f(z) - a| < \epsilon, \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

- If the limit of a function $f(z)$ exists at a point z_0, it is **unique**.
- If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$ then,
$$\lim_{z \to z_0} f(z) = u_0 + iv_0 \iff \lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0.$$

Limit contd....

Note:

- The point z_0 can be approached from **any direction**. If the limit $\lim_{z \to z_0} f(z)$ exists, then $f(z)$ must approach a **unique** limit, no matter how z approaches z_0.
- If the limit $\lim_{z \to z_0} f(z)$ is different for different path of approaches then $\lim_{z \to z_0} f(z)$ does not exists.

Let f, g be complex valued functions with $\lim_{z \to z_0} f(z) = \alpha$ and $\lim_{z \to z_0} g(z) = \beta$.

Then,

- $\lim_{z \to z_0} [f(z) \pm g(z)] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z) = \alpha \pm \beta.$
- $\lim_{z \to z_0} [f(z) \cdot g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z) = \alpha\beta.$
- $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \frac{\alpha}{\beta} \quad (\text{if} \quad \beta \neq 0).$
- $\lim_{z \to z_0} Kf(x) = K \lim_{z \to z_0} f(z) = K\alpha \quad \forall \quad K \in \mathbb{C}.$

Properties of continuous functions

- **Continuity at a point:** A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if for for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon, \quad \text{whenever} \quad |z - z_0| < \delta.$$

  In other words, f is continuous at a point z_0 if the following conditions are satisfied.

  - $\lim_{z \to z_0} f(z)$ exists,
  - $\lim_{z \to z_0} f(z) = f(z_0)$.

- A function f is continuous at z_0 if and only if for every sequence $\{z_n\}$ converging to z_0, the sequence $\{f(z_n)\}$ converges to $f(z_0)$.
- A function f is continuous on D if it is continuous at each and every point in D.
- A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if and only if $u(x, y) = \text{Re }(f(z))$ and $v(x, y) = \text{Im }(f(z))$ are continuous at z_0.

Continuity

Let $f, g : D \subseteq \mathbb{C} \to \mathbb{C}$ be continuous functions at the point $z_0 \in D$. Then

- $f \pm g,\ fg,\ kf\ (k \in \mathbb{C}), \dfrac{f}{g}\ (g(z_0) \neq 0)$ are continuous at z_0.
- Composition of continuous functions is continuous.
- $\overline{f(z)}, |f(z)|, \text{Re}\ (f(z))$ and $\text{Im}\ (f(z))$ are continuous.
- If a function $f(z)$ is continuous and nonzero at a point z_0, then there is a $\delta > 0$ such that $f(z) \neq 0, \ \ \forall\ z \in B(z_0, \delta)$.
- Continuous image of a compact set (closed and bounded set) is compact.