Differentiability

Differentiability

Recall: Let A be a nonempty open subset of $\mathbb{R}$. $x_0 \in A$. Then we say f is differentiable at x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

- **Definition:** Let D be a nonempty open subset of $\mathbb{C}$. $z_0 \in D$. Then f is differentiable at z_0 if the limit

  $$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

  exists. The value of the limit is denoted by $f'(z_0)$ and is called the derivative of f at the point z_0.
- Let $f(z) = z^2$. Then $f(z + h) - f(z) = 2zh + h^2$ and hence the above limit is $2z$. In general, $\frac{d}{dz}(z^n) = nz^{n-1}$, $n \in \mathbb{N}$.
- If $g(z) = \overline{z}$ then the function g is not differentiable anywhere in $\mathbb{C}$. As

  $$\lim_{h \to 0} \frac{g(z + h) - g(z)}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}$$

  does not exist.

- If f is differentiable at z_0 then f is continuous at z_0.

  Proof: Since $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ it follows that

  $$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) + f(z_0) = f(z_0).$$

- Derivative of a constant function is zero. However, converse need not be true.

Suppose f, g be differentiable at z_0 and $\alpha, \beta \in \mathbb{C}$. Then

- $(\alpha f + \beta g)' = \alpha f' + \beta g'$.
- If $h(z) = f(z)g(z)$, then $h'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
- If $f(z) = \frac{g(z)}{h(z)}$ and $h(z_0) \neq 0$, then

  $$f'(z_0) = \frac{g'(z_0)h(z_0) - g(z_0)h'(z_0)}{[h(z_0)]^2}.$$

- **(Chain Rule)** $\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$ whenever all the terms make sense.

Differentiability

Question: Is there any difference between the differentiability in $\mathbb{R}^2$ and $\mathbb{C}$?

- Let $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = |z|^2$. Consider

$$\lim_{h \to 0} \frac{|z_0 + h|^2 - |z_0|^2}{h} = \lim_{h \to 0} \frac{z_0 \bar{h} + \bar{z}_0 h + h\bar{h}}{h} = z_0 \lim_{h \to 0} \frac{\bar{h}}{h} + \bar{z}_0 + \bar{h}.$$

  The above limit exists if and only if $z_0 = 0$. i.e. the function $f(z)$ is complex differentiable only at 0.

- However if we view the same function f as $f : \mathbb{R}^2 \to \mathbb{R}$ i.e. $f(x, y) = x^2 + y^2$ then f is differentiable everywhere on $\mathbb{R}^2$.

Differentiability

Let D be an open subset of $\mathbb{C}$ and $f : D \to \mathbb{C}$ such that

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

Let $z_0 = x_0 + iy_0 \in D$ then

- $u_x(x_0, y_0) = \lim_{h \to 0} \dfrac{u(x_0 + h, y_0) - u(x_0, y_0)}{h}.$
- $u_y(x_0, y_0) = \lim_{k \to 0} \dfrac{u(x_0, y_0 + k) - u(x_0, y_0)}{k}.$

Analogously one can define $v_x(x_0, y_0)$, $v_y(x_0, y_0)$ and higher order partial derivatives of u and v at (x_0, y_0).

Necessary condition for Differentiability

Theorem Suppose that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point $z_0 = (x_0, y_0)$ and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Thus equating the real and imaginary parts we get

$$u_x = v_y, \quad u_y = -v_x, \text{ at } z_0 = x_0 + iy_0 \qquad (\textbf{Cauchy Riemann equations}).$$

Proof. Since f is differentiable at z_0 letting $h = h_1 + ih_2$ tending to 0 in two different paths we get the same limit.

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0 + h_1, y_0) - u(x_0, y_0) + i[v(x_0 + h_1, y_0) - v(x_0, y_0]}{h} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0), \quad [h_1 \to 0, \ h_2 = 0] \end{aligned}$$

Necessary condition for Differentiability

and

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0,y_0+h_2)-u(x_0,y_0)+i[v(x_0,y_0+h_2)-v(x_0,y_0)]}{ih} \\ &= \lim_{h\to 0} \frac{v(x_0,y_0+h_2)-v(x_0,y_0)}{h} - i\lim_{h\to 0} \frac{u(x_0,y_0+h_2)-u(x_0,y_0)}{h} \\ &= v_y(x_0,y_0) - iu_y(x_0,y_0) \quad [h_1=0, h_2\to 0]. \end{aligned}$$

Thus equating the real and imaginary parts of $f'(z_0)$ we get

$u_x(x_0,y_0) = v_y(x_0,y_0),\ \ u_y(x_0,y_0) = -v_x(x_0,y_0),$ (**Cauchy Riemann equations**).

Necessary condition for Differentiability

Summary:

- f is differentiable at $z_0 \Rightarrow$ partial derivatives of u and v exist at z_0 and f satisfies Cauchy Riemann equations.
- The partial derivatives of u and v exist at the point $z_0 = (x_0, y_0)$ but f **DOES NOT** satisfy Cauchy Riemann equations $\implies$ f is **NOT** differentiable at z_0.
- Take $f(z) = |z|^2$. Let $z_0 = (x_0, y_0) \neq (0, 0)$. Here $u(x, y) = x^2 + y^2$ and $V(x, y) = 0$. Then

$$u_x(x_0, y_0) = 2x_0, u_y(x_0, y_0) = 2y_0, v_x(x_0, y_0) = 0 = v_y(x_0, y_0)$$

  f does NOT satisfy Cauchy Riemann equations and hence not differentiable at z_0.
- f satisfies Cauchy Riemann equations at $z_0 \not\Longrightarrow$ f is differentiable at z_0.

Necessary condition for Differentiability

Example: Let

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{(x,\, y) \to (0,\, 0)} \frac{\left(\frac{x^3 - 3xy^2}{x^2 + y^2} + \; i \; \frac{y^3 - 3x^2 y}{x^2 + y^2} \right) - 0}{x + iy - 0}$$

Let z approach 0 along the x-axis. Then, we have

$$\lim_{(x,\, 0) \to (0,\, 0)} \frac{x - 0}{x - 0} = 1 \,.$$

Let z approach 0 along the line $y = x$. This gives

$$\lim_{(x,\, x) \to (0,\, 0)} \frac{-x - ix}{x + ix} = -1 \,.$$

Since the limits are different along two different paths, we conclude that f is not differentiable at the origin.

Necessary condition for Differentiability

$$u_x(0,\ 0) = \lim_{x\to 0} \frac{u(x,\ 0) - u(0,\ 0)}{x - 0} = \lim_{x\to 0} \frac{x - 0}{x} = 1\ .$$

In a similar fashion, one can show that

$$u_y(0,\ 0) = 0, \qquad v_x(0,\ 0) = 0 \qquad \text{and} \qquad v_y(0,\ 0) = 1\ .$$

Hence the function satisfies the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ at the point $z = 0$.

Cauchy-Riemann equation in polar form

- Let $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. The polar form of Cauchy Riemann equation can be obtained as follows:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

- **Result:** Let D be a domain in $\mathbb{C}$. If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is such that $f'(z) = 0$ for all $z \in D$, then f is a constant function.

Sufficient condition for Differentiability

Theorem Let the function $f = u + iv$ be defined on $B(z_0, r)$ such that u_x, u_y, v_x, v_y exist on $B(z_0, r)$ and are continuous at z_0. If u and v satisfies CR equations then $f'(z_0)$ exists and $f'(z_0) = u_x(z_0) + iv_x(z_0)$.

Exercise: Using the above result we can immediately check that the functions

1. $f(x + iy) = x^3 - 3xy^2 + i(3x^2y - y^3)$
2. $f(x + iy) = e^{-y}\cos x + ie^{-y}\sin x$

are differentiable everywhere in the complex plane.