Elementary functions

The Exponential Function

Recall:

- Euler's Formula: For $y \in \mathbb{R}$, $e^{iy} = \cos y + i \sin y$
- and for any $x, y \in \mathbb{R}$, $e^{x+y} = e^x e^y$.

Definition: If $z = x + iy$, then e^z or $\exp(z)$ is defined by the formula

$$e^z = e^{(x+iy)} = e^x(\cos y + i \sin y).$$

Properties of Exponential Function

- $e^{z+w} = e^z e^w, \quad \forall z, w \in \mathbb{C}$. Let $z = x + iy$, $w = s + it$. So

$$\begin{aligned} e^{z+w} &= e^{(x+s)+i(y+t)} = e^{(x+s)}[\cos(y+t) + i \sin(y+t)] \\ &= e^x e^s[(\cos y \cos t - \sin y \sin t) + i(\sin y \cos t + \cos y \sin t)] \\ &= [e^x(\cos y + i \sin y)][e^s(\cos t + i \sin t)] = e^z e^w. \end{aligned}$$

- $e^z \neq 0$, for all $z \in \mathbb{C}$. Look at $|e^z| = |e^x||e^{iy}| = e^x \neq 0$.

The Exponential Function

Properties of Exponential function

- $\frac{\mathrm{d}}{\mathrm{d}\,z}e^z = e^z$. By definition $e^z = e^x\cos y + ie^x\sin y$ satisfies C-R equation on $\mathbb{C}$ and has continuous first order partial derivatives. So e^z is entire and

$$\frac{\mathrm{d}}{\mathrm{d}\,z}e^z = \frac{\mathrm{d}}{\mathrm{d}\,x}e^x\cos y + i\frac{\mathrm{d}}{\mathrm{d}\,x}e^x\sin y = e^z.$$

- e^z is periodic with period $2\pi ni$ for some $n \in \mathbb{Z}$.
  (A function $f : \mathbb{C} \to \mathbb{C}$ is called **periodic** if there is a $w \in \mathbb{C}$ (called a **period**) such that $f(z+w) = f(z)$ for all $z \in \mathbb{C}$.)
- If $w \in \mathbb{C}$ is a period of e^z then $e^{z+w} = e^z$ for all $z \in \mathbb{C}$. In particular for $z = 0$ we have $e^w = 1$. If $w = s + it$ then $e^{it} = 1$, i.e. $t = 2\pi n$ for some $n \in \mathbb{N}$.
- e^z is not injective *unlike* real exponential.
- $\overline{e^z} = e^{\bar{z}}$, $e^0 = 1, |e^z| \leq e^{|z|}$.

The Exponential Function

Mapping Properties of Exponential function:

- $\{(x, y_0) : x \in \mathbb{R}\} \longmapsto \{(r, \theta) : r = e^x, x \in \mathbb{R}, \theta = y_0\}$.
- $\{(x_0, y) : y \in \mathbb{R}\} \longmapsto \{(r, \theta) : r = e^{x_0}, \theta \in \mathbb{R}\}$.
- $\{(x, y) : a \leq x \leq b, c \leq y \leq d\} \longmapsto \{(r, \theta) : e^a \leq r \leq e^b, c \leq \theta \leq d\}$.

Trigonometric Functions

Define

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Properties:

- $\sin^2 z + \cos^2 z = 1.$
- $\sin(z+w) = \sin z \cos w + \cos z \sin w$ and
  $\cos(z+w) = \cos z \cos w - \sin z \sin w$
- $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$, $\sin(z + 2k\pi) = \sin z$,
  $\cos(z + 2k\pi) = \cos z$,.
- $\sin z = 0 \iff z = n\pi$ and $\cos z = 0 \iff z = (n + \frac{1}{2})\pi, \ , \ n \in \mathbb{Z}.$
- $\sin z, \cos z$ are **entire** functions.
- $\frac{d}{dz}(\sin z) = \cos z$, $\frac{d}{dz}(\cos z) = -\sin z.$
- **Prove/Disprove:** $\sin z$ is bounded $\forall\ z \in \mathbb{C}.$

Trigonometric functions

- Define

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \csc z = \frac{1}{\sin z}.$$

- **Hyperbolic Trigonometric functions:** Define
  $\sinh z = \frac{e^z - e^{-z}}{2} \cosh z = \frac{e^z + e^{-z}}{2}$.
- **Properties:**
  - $\sinh z, \cosh z$ are **entire** functions.
  - $\cosh^2 z - \sinh^2 z = 1$.
  - $\sinh(-z) = -\sinh z,\ \cosh(-z) = \cosh z,$
  - $\sinh(z + 2k\pi i) = \sinh z,\ \cosh(z + 2k\pi i) = \cosh z,\ \ k \in \mathbb{Z}.$
  - $\sinh(iz) = i \sin z$ and $\cos(iz) = \cos z$
  - $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$, and
    $\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$ where
    $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$.

Complex Logarithm

Recall:

- e^z is not an **injective** function as $e^{z+2\pi ik} = e^z, \; k \in \mathbb{Z}$.
- e^z is an **onto** function from $\mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Take $w \in \mathbb{C}^*$ then $w = |w|e^{i\theta}$ where $\theta \in (-\pi, \pi]$. If we define $z = \log|w| + i\theta$ then

$$e^z = e^{\log|w|+i\theta} = e^{\log|w|}e^{i\theta} = w.$$

- *If we restrict the domain of the exponential then it becomes injective.* If $H = \{z = x + iy : x \in \mathbb{R}, -\pi < y \leq \pi\}$ then $z \to e^z$ is a **bijective** function from H to $\mathbb{C} \setminus \{0\}$.

Question: What is the inverse of this function?

Complex Logarithm

Definition: For $z \in \mathbb{C}^*$, **define** $\log z = \ln|z| + i \arg z$.

- $\ln|z|$ stands for the real logarithm of $|z|$.
- Since $\arg z = \text{Arg}z + 2k\pi,\ k \in \mathbb{Z}$ it follows that $\log z$ is not well defined as a function. (**multivalued**)
- For $z \in \mathbb{C}^*$, the **principal value** of the logarithm is defined as $\text{Log } z = \ln|z| + i\ \text{Arg}z$.
- $\text{Log} : \mathbb{C}^* \to \{z = x + iy : x \in \mathbb{R}, -\pi < y \leq \pi\}$ is well defined (**single valued**).
- $\text{Log } z + 2k\pi i = \log z$ for some $k \in \mathbb{Z}$.

Complex Logarithm

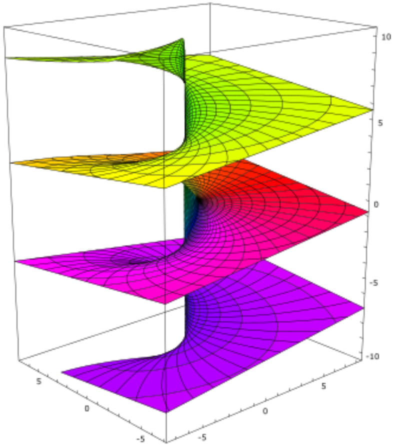
- If $z \neq 0$ then $e^{\text{Log } z} = e^{\ln|z| + i \text{ Arg} z} = z$ (What about Log (e^z)?).
- Suppose x is a positive real number then Log $x = \ln x + i$ Arg$x = \ln x$.
- Log $i = \ln|i| + i\frac{\pi}{2} = \frac{i\pi}{2}$, Log $(-1) = \ln|-1| + i\pi = i\pi$,
  Log $(-i) = \ln|-i| + i\frac{-\pi}{2} = -\frac{i\pi}{2}$, Log $(-e) = 1 + i\pi$ (check!))
- The function Log z is **not continuous** on the **negative real axis** $\mathbb{R}^- = \{z = x + iy : x < 0, y = 0\}$.

  To see this consider the point $z = -\alpha$, $\alpha > 0$. Consider the sequences $\{a_n = \alpha e^{i(\pi - \frac{1}{n})}\}$ and $\{b_n = \alpha e^{i(-\pi + \frac{1}{n})}\}$. Then
  $\lim_{n\to\infty} a_n = z = \lim_{n\to\infty} b_n$ but
  $\lim_{n\to\infty}$ Log $a_n = \lim_{n\to\infty} \ln\alpha + i(\pi - \frac{1}{n}) = \ln\alpha + i\pi$ and
  $\lim_{n\to\infty}$ Log $b_n = \ln\alpha - i\pi$.

Complex Logarithm

- $z \to \text{Log } z$ is **analytic** on the set $\mathbb{C}^* \setminus \mathbb{R}^-$. Let $z = re^{i\theta} \neq 0$ and $\theta \in (-\pi, \pi)$. Then $\text{Log } z = \ln r + i\theta = u(r, \theta) + iv(r, \theta)$ with $u(r, \theta) = \ln r$ and $v(r, \theta) = \theta$. Then $u_r = \frac{1}{r} v_\theta = \frac{1}{r}$ and $v_r = -\frac{1}{r} u_\theta$.
- The identity $\text{Log } (z_1 z_2) = \text{ Log} z_1 + \text{Log } z_2$ is not always valid. However, the above identity is true if and only if $\text{Arg } z_1 + \text{Arg } z_2 \in (-\pi, \pi]$ (why?).
- **Branch of a multiple valued function:** Let F be a multiple valued function defined on a domain D. A function f is said to be a **branch** of the multiple valued function F if in a domain $D_0 \subset D$ if $f(z)$ **is single valued and analytic in** D_0.
- **Branch Cut:** The portion of a line or a curve introduced in order to define a branch of a multiple valued function is called **branch cut**.
- **Branch Point:** Any point that is common to all branch cuts is called a **branch point**.

Complex Logarithm



Complex Exponents

Let $w \in \mathbb{C}$. For any $z \neq 0$, define

$$z^w = exp(w \log z),$$

where "exp" is the exponential function and log is the multiple valued logarithmic function.

- z^w is a multiple valued function.
- $i^i = \exp[i \log i] = \exp[i(\log 1 + i\frac{\pi}{2})] = \exp(-\frac{\pi}{2})$.