Note Title 9/15/2020

Well-ordering principle is logically equivalent to the principle of Mathematical Induction:

(1) Well-ordering principle: Let T be a non-empty subset of $\mathbb{N}$. Then, T has a least element.

(2) Principle of Mathematical Induction: Let $S \subseteq \mathbb{N}$ and $1 \in S$. Suppose that $n \in S \Rightarrow n+1 \in S$. Then $S = \mathbb{N}$.

Proof: (1) $\Rightarrow$ (2): Let $S \subseteq \mathbb{N}$ and $1 \in S$. Suppose that $k \in S \Rightarrow 1+k \in S$. Claim: $S = \mathbb{N}$. We will establish this claim using well-ordering principle.

Suppose $S \neq \mathbb{N}$. Then $\mathbb{N} - S$ is non-empty, and by well-ordering principle, $\mathbb{N} - S$ has a least element, say m.

Since $1 \in S$, so $1 \notin \mathbb{N} - S$. But $m \in \mathbb{N} - S$, so $m \neq 1$.

$\therefore m - 1 \geq 1 \Rightarrow m - 1 \in \mathbb{N}$.

But $m - 1 < m$ and m is the least element of $\mathbb{N} - S$.

$\therefore m - 1 \notin \mathbb{N} - S \Rightarrow m - 1 \in S$.

But the given condition implies that $1 + (m-1) \in S$

$\Rightarrow m \in S$ (which is a contradiction)

$\therefore S = \mathbb{N}$

We next prove that $(2) \Rightarrow (1)$:

Let $S(n)$ be the statement: Any set of natural numbers containing a natural number $\leq n$ has a least element.

Consider the set $A = \{m \in \mathbb{N} \mid S(m) \text{ is true}\}$

Claim: $A = \mathbb{N}$. This will establish well-ordering principle.

(To establish this claim, we will use Mathematical Induction)

We know that 1 is the least natural number. Hence, $S(1)$ is true. $\Rightarrow 1 \in A$. Suppose that $m \in A$, that is,

$S(m)$ is true.

$\Rightarrow$ If X is a subset of $\mathbb{N}$ containing a natural number $\leq m$, then X has a least element.

Let Y be a subset of $\mathbb{N}$ containing a natural number $\leq m+1$.

If Y has no element less than $m+1$, then $m+1$ is the least element of Y. Suppose that $\exists y \in Y$ such that $y < m+1$. Then, $y \leq m$.

Since, we are assuming that $S(m)$ is true, so Y

has a least element. Thus, in any case, Y has a least element.

That is, any subset of $\mathbb{N}$ containing a natural number $\leq m+1$ has a least element.

$\therefore m+1 \in A$.

Thus, $1 \in A$, and $m \in A \Rightarrow m+1 \in A$

By the principle of mathematical induction,

$A = \mathbb{N}$.

That is, $S(n)$ is true for all $n \geq 1$.

This implies the well-ordering principle. #