PROJECTIVE AND INJECTIVE MODULES

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CERTIFICATE

This is to certify that the work contained in this report entitled **"Projective and Injective Modules"** submitted by **Bikash Debnath** (**Roll No: 132123008**) to the Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

Projective and Injective modules arise quite abundantly in nature. For example, all free modules that we know of, are projective modules. Similarly, the group of all rational numbers and any vector space over any field are examples of injective modules. In this thesis, we study the theory of projective and injective modules.

Contents

Chapter 1

Introduction

1.1 Some Basic Definitions

Modules over a ring are a generalization of abelian groups (which are modules over $\mathbb{Z}$).

Definition 1. *Let R be a ring. A (left) R-**module** is an additive abelian group A together with a function $\mu : R \times A \longrightarrow A$ ($\mu(r, a)$ being denoted by ra) such that for all r, $s \in R$ and a, $b \in A$:*

(1) $r(a + b) = ra + rb$.

(2) $(r + s)a = ra + rb$.

(3) $r(sa) = (rs)a$.

If R has an identity element 1_R and

(4) $1_R a = a \quad \forall a \in A$,

*then A is said to be a **unitary** R-**module**.*

Example 1. *Every additive abelian group G is a unitary $\mathbb{Z}$-module, with na ($n \in \mathbb{Z}$, $a \in G$) defined by $na = a + a + ... + a$ (n times).*

1.1.1 R- Module Homomorphism

Definition 2. *Let A and B be modules over a ring R. A function $f : A \longrightarrow B$ is an R-**module homomorphism** provided that for all $a,\ c \in A$ and $r \in R$:*

(1) $f(a + c) = f(a) + f(c)$ and

(2) $f(ra) = rf(a)$.

Definition 3. *Let $\{M_i\}_{i\in I}$ be a family of R-modules. Their **direct sum** $\oplus_{i\in I}M_i$ is the set of all tuples $(x_i)_{i\in I}$ such that $x_i \in M_i$ for all $i \in I$ and all but finitely many x_i are 0. This set $\oplus_{i\in I}M_i$ has a natural structure of an R-module given by:*

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$$

$$a(x_i)_{i\in I} = (ax_i)_{i\in I}$$

for all $a \in R$ and for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \oplus_{i\in I}M_i$.

Definition 4. *Let $\{M_i\}_{i\in I}$ be a family of R-modules. Their **direct product** $\Pi_{i\in I}M_i$ is the set of all tuples $(x_i)_{i\in I}$ such that $x_i \in M_i$ for all $i \in I$. This set $\Pi_{i\in I}M_i$ has a natural structure of an R-module given by:*

$$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$$

$$a(x_i)_{i\in I} = (ax_i)_{i\in I}$$

for all $a \in R$ and for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \Pi_{i\in I}M_i$.

The following theorems follow easily from the definitions of direct sum and direct product of modules. So, we omit their proofs.

Theorem 1. *If R is a ring, $\{A_i | i \in \mathbb{I}\}$ a family of R- modules, C an R-module, and $\{\phi_i : C \longrightarrow A_i | i \in \mathbb{I}\}$ a family of R-module homomorphisms, then there is a unique R-module homomorphism $\phi : C \longrightarrow \prod_{i\in\mathbb{I}} A_i$ such that $\pi_i\phi = \phi_i \quad \forall i \in \mathbb{I}$. $\prod_{i\in\mathbb{I}} A_i$ is uniquely determined up to isomorphism by this property.*

Theorem 2. *If R is a ring, $\{A_i | i \in \mathbb{I}\}$ a family of R- modules, D an R module, and $\{\psi_i : A_i \longrightarrow D | i \in \mathbb{I}\}$ a family of R-module homomorphisms, then there is a unique R-module homomorphism $\psi : \sum_{i \in \mathbb{I}} A_i \longrightarrow D$ such that $\psi \iota_i = \psi_i \quad \forall i \in \mathbb{I}$. $\sum_{i \in \mathbb{I}} A_i$ is uniquely determined up to isomorphism by this property.*

1.1.2 Exact Sequences

Definition 5. *A pair of module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, is said to be* ***exact*** *at B provided* ***Im*** *f =* ***Ker*** *g.*

Definition 6. *A finite sequence of module homomorphisms, $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} ... \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$ is* ***exact*** *provided* ***Im*** *f_i =* ***Ker*** *f_{i+1} for $i = 1, 2, 3 ... n-1$.*

Definition 7. *An infinite sequence of module homomorphisms, $... \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} ...$ is* ***exact*** *provided* ***Im*** *f_i =* ***Ker*** *f_{i+1} for all $i \in \mathbb{Z}$.*

Remark 1. *$0 \longrightarrow A \xrightarrow{f} B$ is exact sequence of module homomorphism iff f is module monomorphism.*
Similarly, $B \xrightarrow{g} C \longrightarrow 0$ is exact sequence of module homomorphism iff g is module epimorphism.
If $A \xrightarrow{f} B \xrightarrow{g} C$ is exact then $gf = 0$.

Definition 8. *An exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called a* ***short exact sequence****.*

Note 1. *In the above definition, f is monomorphism and g is epimorphism.*

Lemma 1. ***The Short five lemma:*** *Let R be a ring and*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

a commutative diagram of R-modules and R-module homomorphisms, such that each row is a exact sequence. Then

(1) α, γ *monomorphisms* $\Rightarrow \beta$ *is a monomorphism;*

(2) α, γ *epimorphisms* $\Rightarrow \beta$ *is a epimorphism;*

(3) α, γ *isomorphisms* $\Rightarrow \beta$ *is a isomorphism.*

Proof. Commutativity gives : $\beta f = f'\alpha$ and $\gamma g = g'\beta$.

`proof of (1):` Let $b \in B$ and suppose $\beta(b) = 0$; we must show that $b = 0$. By commutativity we have

$$\gamma g(b) = g'\beta(b) = g'(0) = 0.$$

$\Rightarrow g(b) = 0$, since γ is monomorphism.
By exactness of the top row at B, we have $b \in Ker\ g = Im\ f$, say $b = f(a)$, $a \in A$. By commutativity,

$$f'\alpha(a) = \beta f(a) = \beta(b) = 0.$$

By exactness of the bottom row at A', f' is a monomorphism. Hence $\alpha(a) = 0$. But α is a monomorphism; therefore $a = 0$ and hence $b = f(a) = f(0) = 0$. Thus β is a monomorphism.

`proof of (2):` Let $b' \in B'$. Then $g'(b') \in C'$; since γ is an epimorphism $g'(b') = \gamma(c)$ for some $c \in C$. By exactness of the top row at C, g is an epimorphism; hence $c = g(b)$ for some $b \in B$. By commutativity,

$$g'\beta(b) = \gamma g(b) = \gamma(c) = g'(b').$$

Thus $g'[\beta(b) - b'] = 0$ and $\beta(b) - b' \in Ker\ g' = Im\ f'$ by exactness, say $f'(a') = \beta(b) - b'$, $a' \in A'$. Since α is an epimorphism, $a' = \alpha(a)$ for some $a \in A$. Consider $b - f(a) \in B$.

$$\beta[b - f(a)] = \beta(b) - \beta(f(a)).$$

By commutativity, $\beta f(a) = f'\alpha(a) = f'(a') = \beta(b) - b'$; hence

$$\beta[b - f(a)] = \beta(b) - \beta f(a) = \beta(b) - (\beta(b) - b') = b'$$

and β is an epimorphism.

`proof of (3):` It is an immediate consequence of `(1)` and `(2)`.

$\square$

1.1.3 Isomorphic Short Exact sequences

Definition 9. *Two short exact sequences are said to be **isomorphic** if there is a commutative diagram of module homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

such that f, g and h are isomorphisms. In this case it is easy to verify that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u, "f^{-1}"] & B' \arrow[r] \arrow[u, "g^{-1}"] & C' \arrow[r] \arrow[u, "h^{-1}"] & 0
\end{tikzcd}$$

(with the same horizontal maps) is also commutative.

Theorem 3. *Let R be ring and $0 \longrightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \longrightarrow 0$ be a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent.*

(1) There is an R-module homomorphism $h : A_2 \longrightarrow B$ with $gh = I_{A_2}$;

(2) There is an R-module homomorphism $k : B \longrightarrow A_1$ with $kf = I_{A_1}$;

(3) The given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence $0 \longrightarrow A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \longrightarrow 0$; in particular $B \cong A_1 \oplus A_2$.

Proof. $\underline{\bullet(1) \Rightarrow (2)}$

By the `Theorem 2`, the homomorphisms f and h induce a unique module homomorphism $\phi : A_1 \oplus A_2 \longrightarrow B$, given by $(a_1, a_2) \longmapsto f(a_1) + h(a_2)$.

Here,

$$\begin{array}{lll} \psi_1 = f : A_1 \longrightarrow B \quad \text{and} \quad \iota_1 : A_1 \longrightarrow A_1 \oplus A_2 \quad \text{by} \quad a_1 \longmapsto (a_1, 0) \\ \psi_2 = h : A_2 \longrightarrow B \quad \text{and} \quad \iota_2 : A_2 \longrightarrow A_1 \oplus A_2 \quad \text{by} \quad a_2 \longmapsto (0, a_2). \end{array}$$

By the `Theorem 2`, $\exists$ a unique R-module homomorphism $\phi : A_1 + A_2 \longrightarrow B$ such that $\phi\iota_1 = f$ and $\phi\iota_2 = h$.
$\phi\iota_1 : A_1 \longrightarrow B, \quad \phi\iota_1(a_1) = f(a_1)$
$\phi\iota_2 : A_2 \longrightarrow B, \quad \phi\iota_2(a_2) = f(a_2)$
and,
$\phi : A_1 + A_2 \longrightarrow B$, therefore

$$\begin{aligned} \phi(a_1, a_2) &= \phi(a_1, 0) + \phi(0, a_2) \\ &= \phi(\iota_1(a_1)) + \phi(\iota_2(a_2)) \\ &= \phi(\iota_1(a_1)) + \phi(\iota_2(a_2)) \\ &= f(a_1) + h(a_2). \end{aligned}$$

Now the diagram

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "\iota_1"] \arrow[d, "1_{A_1}"'] & A_1 \oplus A_2 \arrow[r, "\pi_2"] \arrow[d, "\varphi"] & A_2 \arrow[r] \arrow[d, "1_{A_2}"] & 0 \\ 0 \arrow[r] & A_1 \arrow[r, "f"] & B \arrow[r, "g"] & A_2 \arrow[r] & 0 \end{tikzcd}$$

is commutative.
To show that the diagram is commutative: Here, $gf = 0$ and $gh = I_{A_2}$.
Now, to show that:

$$\begin{aligned} \phi\iota_1 &= fI_{A_1} \\ \textit{and} \quad g\phi &= I_{A_2}\pi_2. \end{aligned}$$

Now, let $a_1 \in A_1, a_2 \in A_2$ and $(a_1, a_2) \in A_1 \oplus A_2$.
Therefore, $\phi\iota_1(a_1) = \phi(a_1, 0) = f(a_1)$.
and $fI_{A_1}(a_1) = f(a_1)$.
so, $\phi\iota_1 = f$.

Now,

$$\begin{aligned} g\phi(a_1, a_2) &= g(f(a_1) + h(a_2)) \\ &= (gf)(a_1) + (gh)(a_2) \\ &= 0 + I_{A_2}(a_2) \\ &= a_2. \end{aligned}$$

So, the diagram is commutative.

Therefore, by the `Short Five Lemma` ϕ is an isomorphism.

<u>$\bullet$(2) $\Rightarrow$ (3)</u>

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "f"] \arrow[d, "1_{A_1}"'] & B \arrow[r, "g"] \arrow[d, "\psi"] & A_2 \arrow[r] \arrow[d, "1_{A_2}"] & 0 \\ 0 \arrow[r] & A_1 \arrow[r, "\iota_1"] & A_1 \bigoplus A_2 \arrow[r, "\pi_2"] & A_2 \arrow[r] & 0 \end{tikzcd}$$

This diagram is commutative.
Where ψ is the module homomorphism given by $\psi(b) = (k(b), g(b))$.
Hence the `Short Five Lemma` implies ψ is an isomorphism.

<u>$\bullet$(3) $\Rightarrow$ (1), (2)</u>

Given a commutative diagram with exact rows and ϕ an isomorphism:

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, shift left, "\iota_1"] \arrow[d, "1_{A_1}"'] & A_1 \bigoplus A_2 \arrow[l, shift left, dashed, "\pi_1"] \arrow[r, shift left, "\pi_2"] \arrow[d, "\varphi"] & A_2 \arrow[l, shift left, dashed, "\iota_2"] \arrow[r] \arrow[d, "1_{A_2}"] & 0 \\ 0 \arrow[r] & A_1 \arrow[r, "f"] & B \arrow[r, "g"] & A_2 \arrow[r] & 0, \end{tikzcd}$$

define $h : A_2 \longrightarrow B$ to be $\phi\iota_2$ and $k : B \longrightarrow A_1$ to be $\pi_1\phi^{-1}$. Use the commutativity of the diagram and the facts $\pi_i\iota_i = I_{A_i}$ and $\phi^{-1}\phi = I_{A_1 \oplus A_2}$ to show that $kf = I_{A_1}$ and $gh = I_{A_2}$.

$\square$

Note 2. *A short exact sequence that satisfies the equivalent conditions of the* `Theorem 3` *is said to be* ***split*** *or* ***split exact sequence.***

1.1.4 Free Module

Theorem 4. *Let R be a ring with identity. The following conditions on a unitary R-module are equivalent* ***:***

(a) F has a non empty basis;

(b) F is isomorphic to direct sum of a family of cyclic R-module to R;

(c) F is R module isomorphic to a direct sum of copies of the left R module R;

(d) there exists a nonempty set X and a function $\iota : X \longrightarrow F$ with the following property: given any unitary R-module A and function $f : X \longrightarrow A$, there exist a unique R-module homomorphism $\bar{f} : F \longrightarrow A$ such that $\bar{f}\iota = f$.

Proof. Omitted. □

Definition 10. *A unitary module F over a ring R with identity, which satisfies the equivalent conditions of* `Theorem 4` *is called a* ***free R-module*** *on the set X.*

Example 2. *Let R any ring and $\mathbb{I}$ be an indexing set. Then $\oplus_{i\in\mathbb{I}} R_i$ where each R_i is isomorphic to R is an example of free module.*

Corollary 4.1. *Every (unitary) module A over a ring R (with identity) is the homomorphic image of a free R-module F.*

Proof. Let X be a set of generators of A and F be the free R-module on the set X. Then the inclusion map $X \longrightarrow A$ induces an R-module homomorphism $\bar{f} : F \longrightarrow A$ such that $X \subset \operatorname{Im} \bar{f}$ (By `Theorem 4`). Since X generates A, we must have $\operatorname{Im} \bar{f} = A$. □

Chapter 2

Projective Modules

2.1 Projective modules : Definition

Definition 11. *A module P over a ring R is said to be* ***projective*** *if given any diagram of R-module homomorphisms*

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

with bottom row exact (that is, g an epimorphism), $\exists$ an R-module homomorphism $h : P \longrightarrow A$ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, "h"'] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

is commutative (that is, $gh = f$).

Theorem 5. *Every free module F over a ring R with identity is projective.*

Proof. Assume that we are given a diagram of homomorphisms of unitary R-modules:

$$\begin{tikzcd} & F \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

with g an epimorphism and F a free R-module on the set X ($\iota : X \longrightarrow F$). For each $x \in X$, $f(\iota(x)) \in B$. Since g is an epimorphism, there exists $a_x \in A$ with $g(a_x) = f(\iota(x))$. Since F is free, the map $X \longrightarrow A$ given by $x \mapsto a_x$ induces an R-module homomorphism $h : F \longrightarrow A$ such that $h(\iota(x)) = a_x$ for all $x \in X$. Consequently, $gh\iota(x) = g(a_x) = f\iota(x)$ for all $x \in X$ so that $gh\iota = f\iota : X \longrightarrow B$. By the uniqueness part of the `Theorem 4` we have $gh = f$. Therefore F is projective.

□

Corollary 5.1. *Every module A over a ring R is the homomorphic image of a projective R-module.*

Proof. It directly follows from the `Corollary 4.1` and `Theorem 5`. □

Theorem 6. *Let R be a ring. The following conditions on an R-module P are equivalent.*

(1) P is projective;

(2) every short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$ is split exact (hence $B \cong A \oplus P$);

(3) there is a free module F and an R-module K such that $F \cong K \oplus P$.

Proof. • <u>(1) ⇒ (2)</u>

Consider the diagram

$$\begin{tikzcd} & P \arrow[d, "1_P"] & \\ B \arrow[r, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

with bottom row exact by the hypothesis. Since P is projective there is an R-module homomorphism $h : P \longrightarrow B$ such that $gh = 1_p$. Therefore, the short exact sequence $0 \longrightarrow A \xrightarrow{f} B \underset{h}{\overset{g}{\rightleftarrows}} P \longrightarrow 0$ is split exact by `Theorem 3` and $B \cong A \oplus P$.

• <u>(2) $\Rightarrow$ (3)</u>

By `Corollary 4.1` there is free R-module F and an epimorphism $g : F \longrightarrow P$. If $K = \text{Ker } g$, then $0 \longrightarrow K \underset{\iota}{\overset{\subset}{\to}} F \overset{g}{\to} P \longrightarrow 0$ is exact. By hypothesis the sequence splits so that $F \cong K \oplus P$ by `Theorem 3`.

• <u>(3) $\Rightarrow$ (1)</u>

Let π be the composition $F \cong K \oplus P \longrightarrow P$ where the second map is the canonical projection. Similarly let ι be the composition $P \longrightarrow K \oplus P \cong F$ with the first map the canonical injection. Given a diagram of R-module homomorphisms

$$\begin{tikzcd}
 & P \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

with exact bottom row, consider the diagram

$$\begin{tikzcd}
 & F \arrow[d, shift right, "\pi"'] & \\
 & P \arrow[u, shift right, "\iota"'] \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

Since F is projective by `Theorem 5`, there is an R-module homomorphism $h_1 : F \longrightarrow A$ such that $gh_1 = f\pi$.
Let $h = h_1\iota : P \longrightarrow A$. Then $gh = gh_1\iota = (f\pi)\iota = f(\pi\iota) = f1_P = f$. Therefore, P is projective.

$\square$

Example 3. ***Projective but not free:*** *If $R = \mathbb{Z}_6$, then $\mathbb{Z}_3$ and $\mathbb{Z}_2$ are $\mathbb{Z}_6$-modules and there is $\mathbb{Z}_6$-module isomorphism $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Hence both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules that are not free $\mathbb{Z}_6$-modules.*

Proposition 1. *Let R be a ring. A direct sum of R-modules $\sum_{i \in \mathbb{I}} P_i$ is projective iff each P_i is projective.*

Proof. Suppose $\sum P_i$ is projective. Since the proof of (3) $\Rightarrow$ (1) in `Theorem-6` uses only the fact that F is projective, it remains valid with $\sum_{i \in \mathbb{I}} P_i$, $\sum_{i \neq j} P_i$ and P_j in place of F, K and P respectively. The converse is proved by similar techniques using the diagram

$$\begin{tikzcd}
& P_j \arrow[d, shift right, "\iota_j"'] & \\
& \sum P_i \arrow[u, shift right, "\pi_j"'] \arrow[d, "f"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0
\end{tikzcd}$$

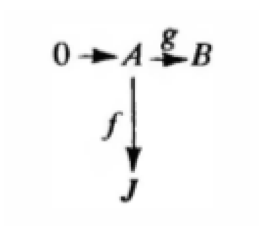
If each P_j is projective, then for each j there exists $h_j : P_j \longrightarrow A$ such that $gh_j = f\iota_j$. By `Theorem 2` there is a unique homomorphism $h : \sum P_i \longrightarrow A$ with $h\iota_j = h_j$ for every j and we also have $gh = f$. $\square$

Chapter 3

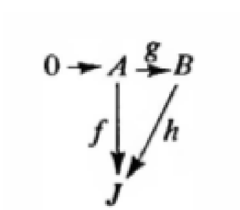
Injective Modules

3.1 Injective modules : Definition

Definition 12. *A module J over a ring is said to be **injective** if given any diagram of R-module homomorphisms*



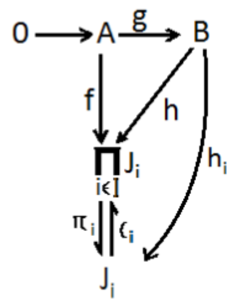
with top row exact (that is, g is a monomorphism), there exists an R-module homomorphism $h : B \longrightarrow J$ such that the diagram



is commutative (that is $hg = f$).

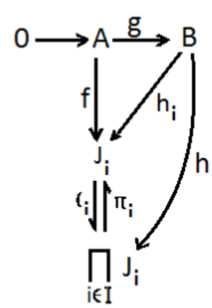
Proposition 2. *A direct product of R-modules $\prod_{i \in \mathbb{I}} J_i$ is injective iff J_i is injective for every $i \in \mathbb{I}$.*

Proof. Suppose that J_i is injective $\forall i \in \mathbb{I}$.
Now in this diagram



we have to find h. Since J_i is injective $\exists\ h_i : B \longrightarrow J_i$ such that $h_i g = \pi_i f$.
Define $h : B \longrightarrow \prod_{i\in\mathbb{I}} J_i$ to be $h(b) := (h_i(b))_{i\in\mathbb{I}} = (h_1(b), h_2(b), ...)$.
Now it is very easy to check that, $hg = f$.

Conversely, suppose that $\prod_{i\in\mathbb{I}} J_i$ is injective. To show that, J_i is injective for each $i \in \mathbb{I}$. Now in this diagram



we have $hg = \iota_i f$.
We have to find h_i. Define $h_i : B \longrightarrow J_i$ to be $h_i = \pi_i h$.
Now it is very easy to check that, $h_i g = f \quad \forall i \in \mathbb{I}$.
Here, $h_i g = \pi_i h g = \pi_i \iota_i f = I_{J_i} f = f \quad \forall i \in \mathbb{I}$.

$\square$

Lemma 2. ***<u>Baer's Criterion:</u>*** *Let R be a ring with identity. A unitary R-module J is injective if and only if for every left ideal L of R, any R-module homomorphism $L \longrightarrow J$ may be extended to an R-module homomorphism $R \longrightarrow J$.*

Proof. To say that $f : L \longrightarrow J$ may be extended to R means there is a homomorphism $h : R \longrightarrow J$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\subset"] \arrow[d, "f"'] & R \arrow[dl, "h"] \\ & J & \end{tikzcd}$$

is commutative. Clearly, such an h always exists if J is injective. Conversely, suppose J has the stated extension property and suppose we are given a diagram of module homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \\ & J & \end{tikzcd}$$

with top row exact. To show that J is injective we must find a homomorphism $h : B \longrightarrow J$ with $hg = f$. Let S be the set of all R-module homomorphisms $h : C \longrightarrow J$, where $\text{Im } g \subset C \subset B$. S is non empty since $fg^{-1} : \text{Im } g \longrightarrow J$ is an element of S (g is a monomorphism). Partially order S by extension : $h_1 \leq h_2$ iff $\text{Dom } h_1 \subset \text{Dom } h_2$ and $h_2 \mid \text{Dom } h_1 = h_1$. We can verify that the hypotheses of `Zorn's Lemma` are satisfied and conclude that S contains a maximal element $h : H \longrightarrow J$ with $hg = f$. We shall complete the proof by showing $H = B$.

If $H \neq B$ and $b \in B - H$, then $L = \{r \in R \mid rb \in H\}$ is left ideal of R. The map $L \longrightarrow J$ given by $r \mapsto h(rb)$ is a well- defined R-module homomorphism. By the hypothesis there is a R-module homomorphism $k : R \longrightarrow J$ such that $k(r) = h(rb)$ for all $r \in L$. Let $c \in k(1_R)$ and define a map $\bar{h} : H + Rb \longrightarrow J$ by $a + rb \mapsto h(a) + rc$. We claim that $\bar{h}$ is well-defined. For if $a_1 + r_1 b = a_2 + r_2 b \in H + Rb$, then $a_1 - a_2 = (r_2 - r_1)b \in H \bigcap Rb$. Hence $r_2 - r_1 \in L$ and $h(a_1) - h(a_2) = h(a_1 - a_2) = h((r_2 - r_1)b) = k(r_2 - r_1) = (r_2 - r_1)k(1_R) = (r_2 - r_1)c$. Therefore, $\bar{h} : R + Rb \longrightarrow J$ is an R-module homomorphism that is an element of the set S. This contradicts the maximality of h since $b \notin H$ and hence $H \subsetneqq H + Rb$. Therefore, $H = B$ and J is injective. $\square$

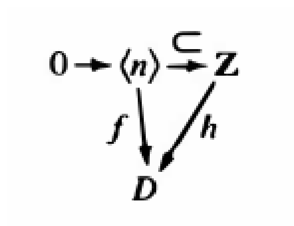
3.1.1 Divisible Group

Definition 13. *An abelian group D is said to be **divisible** if given any $y \in D$ and $0 \neq n \in \mathbb{Z}$, there exists $x \in D$ such that $nx = y$.*

For example, the additive group $\mathbb{Q}$ is divisible, but $\mathbb{Z}$ is not. The factor group $\mathbb{Q}/\mathbb{Z}$ is also divisible group.

Lemma 3. *An abelian group D is divisible iff D is an injective (unitary) $\mathbb{Z}$-module.*

Proof. If D is injective, $y \in D$ and $0 \neq n \in \mathbb{Z}$, let $f : \langle n \rangle \longrightarrow D$ be the unique homomorphism determined by $n \mapsto y$; ($\langle n \rangle$ is a free $\mathbb{Z} - module$). Since D is injective, there is a homomorphism $h : \mathbb{Z} \longrightarrow D$ such that the diagram



is commutative. If $x = h(1)$, then $nx = nh(1) = h(n) = f(n) = y$. Therefore, D is divisible. To prove the converse note that the only left ideals of $\mathbb{Z}$ are the cyclic groups $\langle n \rangle, n \in \mathbb{Z}$. If D is divisible and $f : \langle n \rangle \longrightarrow D$ is a homomorphism, then there exists $x \in D$ with $nx = f(n)$. Define $h : \mathbb{Z} \longrightarrow D$ by $1 \mapsto x$ and verify that h is a homomorphism that extends f. Therefore, D is injective by `Lemma 2`.

$\square$

Remark 2. *The rationals $\mathbb{Q}$ (with addition) form an injective abelian group (i.e. an injective $\mathbb{Z}$-module). The factor group $\mathbb{Q}/\mathbb{Z}$ is also injective $\mathbb{Z}$-module.*

Lemma 4. *Every abelian group A may be embedded in a divisible abelian group.*

Proof. There is a free $\mathbb{Z}$-module F and an epimorphism $F \longrightarrow A$ with kernel K so that $F/K \cong A$. Since F is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$, F may be embedded in a direct sum D of copies of the rationals $\mathbb{Q}$. But D is a divisible group by `Proposition 2, Lemma 3`. If $f : F \longrightarrow D$ is the embedding monomorphism, then f induces an isomorphism $F/K \cong f(F)/f(K)$. Thus the composition $A \cong F/K \cong f(F)/f(K) \subset D/f(K)$ is a monomorphism. But $D/f(K)$ is divisible since it is the homomorphic image of a divisible group. $\square$

Lemma 5. *If J is a divisible abelian group and R is a ring with identity, then $Hom_{\mathbb{Z}}(R, J)$ is an injective left R-module.*

Proof. By `Lemma 2` it suffices to show that for each left ideal L of R, every R-module homomorphism $f : L \longrightarrow Hom_{\mathbb{Z}}(R, J)$ may be extended to an R-module homomorphism $h : R \longrightarrow Hom_{\mathbb{Z}}(R, J)$. The map $g : L \longrightarrow J$ given by $g(a) = [f(a)](1_R)$ is a group homomorphism. Since J is an injective $\mathbb{Z}$-module by `Lemma 3` and we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\subset"] \arrow[d, "g"'] & R \\ & J & \end{tikzcd}$$

there is group homomorphism $\bar{g} : R \longrightarrow J$ such that $\bar{g} \mid L = g$. Define $h : R \longrightarrow Hom_{\mathbb{Z}}(R, J)$ by $r \mapsto J$, where $h(r) : R \longrightarrow J$ is the map given by $[h(r)](x) = \bar{g}(xr) \qquad (x \in R)$. h is well-defined function (that is, each $h(r)$ is a group homomorphism $R \longrightarrow J$) and h is group homomorphism $R \longrightarrow Hom_{\mathbb{Z}}(R, J)$. If $s, r, x \in R$, then

$$h(sr)(x) = \bar{g}(x(sr)) = \bar{g}((xs)r) = h(r)(xs).$$

By the definition of the R-module structure of $Hom_{\mathbb{Z}}(R, J), h(r)(xs) = [sh(r)](x)$, whence $h(sr) = sh(r)$ and h is an R-module homomorphism. Finally suppose $r \in L$ and $x \in R$. Then $xr \in L$ and

$$h(r)(x) = \bar{g}(xr) = g(xr) = [f(xr)](1_R).$$

Since f is an R-module homomorphism and $Hom_{\mathbb{Z}}(R, J)$ an R-module,

$$[f(xr)](1_R) = [xf(r)](1_R) = f(r)(1_R x) = f(r)(x).$$

Therefore, $h(r) = f(r)$ for $r \in L$ and h is an extension of f.

$\square$

Proposition 3. *Every unitary module A over a ring R with identtity may be embedded in an injective R-module.*

Proof. Since A is an abelian group, there is a divisible group J and a group monomorphism $f : A \longrightarrow J$ by `Lemma 4`. The map $\bar{f} : Hom_{\mathbb{Z}}(R, A) \longrightarrow Hom_{\mathbb{Z}}(R, J)$ given on $g \in Hom_{\mathbb{Z}}(R, A)$ by $\bar{f}(g) = fg \in Hom_{\mathbb{Z}}(R, J)$ is easily seen to be an R-module monomorphism. Since every R-module homomorphism is a $\mathbb{Z}-$module homomorphism, we have $Hom_R(R, A) \subset Hom_{\mathbb{Z}}(R, A)$. In fact, it is easy to see that $Hom_R(R, A)$ is an R-submodule of $Hom_{\mathbb{Z}}(R, A)$. Finally, the map $A \longrightarrow Hom_R(R, A)$ given by $a \mapsto f_a$, where $f_a(r) = ra$, is an R-module monomorphism (in fact it is an isomorphism). Composing these maps yeilds an R-module monomorphism

$$A \longrightarrow Hom_R(R, A) \xrightarrow{\subset} Hom_{\mathbb{Z}}(R, A) \xrightarrow{\bar{f}} Hom_{\mathbb{Z}}(R, J).$$

Since $Hom_{\mathbb{Z}}(R, J)$ is an injective R-module by `Lemma 5`, we have embedded A in an injective.

$\square$

Proposition 4. *Let R be a ring with identity. The following conditions on a unitary R-module J are equivalent.*

(1) J is injective ;

(2) every short exact sequence $0 \longrightarrow J \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact (hence $B \cong J \oplus C$);

(3) J is a direct summand of any module B of which it is a submodule.

Proof. (1) $\Rightarrow$ (2): Dualize the proof (1) $\Rightarrow$ (2) of `Theorem 6`.

(2) $\Rightarrow$ (3): Since the sequence $0 \longrightarrow J \xrightarrow{\subset} B \xrightarrow{\pi} B/J \longrightarrow 0$ is split exact, there is a homomorphism $g : B/J \longrightarrow B$ such that $\pi g = 1_{B/J}$. By `Theorem 3`, (1) $\Rightarrow$ (3) there is an isomorphism $J \oplus B/J \cong B$ given by $(x, y) \mapsto x + g(y)$. It folows easily that B is the internal direct sum J and $g(B/J)$.

(3) $\Rightarrow$ (1): It follows from `Proposition 3` that J is a submodule of an injective module Q. `Proposition 2` and (3) imply that J is injective.

$\square$

Example 4. ***Injective module:*** *Given a field K, every K vector space W is an injective K-module. Reason: if W is a subspace of V, we can find a basis of W and extend it to a basis of V. The new extended basis vectors span a subspace S of V and V is the internal direct sum of W and S. The proof follows from the equivalence* $(1) \Leftrightarrow (3)$ *of* `Proposition 4`*.*

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