Ordinary Differential Equations

(MA102 Mathematics II)

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First order ODE s

We will now discuss different methods of solutions of first order ODEs. The first type of such ODEs that we will consider is the following:

Definition

Separable variables: A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called **separable** or to have **separable variables**.

Such ODEs can be solved by direct integration:
Write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)dx$ and then integrate both sides!

Example

$e^x \frac{dy}{dx} = e^{-y} + e^{-2x-y}$
This equation can be rewritten as $\frac{dy}{dx} = e^{-x}e^{-y} + e^{-3x-y}$, which is the same as $\frac{dy}{dx} = e^{-y}(e^{-x} + e^{-3x})$. This equation is now in separable variable form.

Losing a solution while separating variables

Some care should be exercised in separating the variables, since the variable divisors could be zero at certain points. Specifically, if r is a zero of the function $h(y)$, then substituting $y = r$ in the ODE $\frac{dy}{dx} = g(x)h(y)$ makes both sides of the equation zero; in other words, $y = r$ is a constant solution of the ODE $\frac{dy}{dx} = g(x)h(y)$. But after variables are separated, the left hand side of the equation $\frac{dy}{h(y)} = g(x)dx$ becomes undefined at r. As a consequence, $y = r$ might not show up in the family of solutions that is obtained after integrating the equation $\frac{dy}{h(y)} = g(x)dx$.

Recall that such solutions are called Singular solutions of the given ODE.

Example

Observe that the constant solution $y \equiv 0$ is lost while solving the IVP $\frac{dy}{dx} = xy; y(0) = 0$ by separable variables method.

First order linear ODEs

Recall that a **first order linear ODE** has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{1}$$

Definition

A first order linear ODE (of the above form (1)) is called **homogeneous** if $g(x) = 0$ and **non-homogeneous** otherwise.

Definition

By dividing both sides of equation (1) by the leading coefficient $a_1(x)$, we obtain a more useful form of the above first order linear ODE, called the **standard form**, given by

$$\frac{dy}{dx} + P(x)y = f(x). \tag{2}$$

Equation (2) is called the **standard form** of a first order linear ODE.

Theorem

Theorem

Existence and Uniqueness: *Suppose* $a_1(x), a_0(x), g(x) \in C((a,b))$ *and* $a_1(x) \neq 0$ *on* (a,b) *and* $x_0 \in (a,b)$. *Then for any* $y_0 \in \mathbb{R}$, *there exists a unique solution* $y(x) \in C^1((a,b))$ *to the IVP*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x); y(x_0) = y_0.$$

Solving a first order linear ODE

Steps for solving a first order linear ODE:

(1) Transform the given first order linear ODE into a first order linear ODE in standard form $\frac{dy}{dx} + P(x)y = f(x)$.

(2) Multiply both sides of the equation (in the standard form) by $e^{\int P(x)dx}$. *Then the resulting equation becomes*

$$\frac{d}{dx}[ye^{\int P(x)dx}] = f(x)e^{\int P(x)dx} \qquad (3).$$

(3) Integrate both sides of equation (3) *to get the solution.*

Example

Solve $x\frac{dy}{dx} - 4y = x^6e^x$.

The standrad form of this ODE is $\frac{dy}{dx} + (\frac{-4}{x})y = x^5e^x$. Then multiply both sides of this equation by $e^{\int \frac{-4}{x}dx}$ and integrate.

Differential of a function of 2 variables

Definition

Differential of a function of 2 variables: If $f(x,y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential df is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

In the special case when $f(x,y) = c$, where c is a constant, we have $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Therefore, we have $df = 0$, or in other words,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

So given a one-parameter family of functions $f(x,y) = c$, we can generate a first order ODE by computing the differential on both sides of the equation $f(x,y) = c$.

Exact differential equation

Definition

A differential expression $M(x,y)dx + N(x,y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined on R. A first order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact equation** if the expression on the left hand side is an exact differential.

Example: 1) $x^2y^3dx + x^3y^2dy = 0$ is an exact equation since $x^2y^3dx + x^3y^2dy = d(\frac{x^3y^3}{3})$.
2) $ydx + xdy = 0$ is an exact equation since $ydx + xdy = d(xy)$.
3) $\frac{ydx-xdy}{y^2} = 0$ is an exact equation since $\frac{ydx-xdy}{y^2} = d(\frac{x}{y})$.

Criterion for an exact differential

Theorem

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition for $M(x, y)dx + N(x, y)dy$ to be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example

Solve the ODE $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$.
This equation can be expressed as $M(x, y)dx + N(x, y)dy = 0$ where $M(x, y) = 3x^2 + 4xy$ and $N(x, y) = 2x^2 + 2y$. It is easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$. Hence the given ODE is exact.
We have to find a function f such that $\frac{\partial f}{\partial x} = M = 3x^2 + 4xy$ and $\frac{\partial f}{\partial y} = N = 2x^2 + 2y$. Now $\frac{\partial f}{\partial x} = 3x^2 + 4xy \Rightarrow f(x, y) = \int (3x^2 + 4xy)dx = x^3 + 2x^2y + \phi(y)$ for some function $\phi(y)$ of y.
Again $\frac{\partial f}{\partial y} = 2x^2 + 2y$ and $f(x, y) = x^3 + 2x^2y + \phi(y)$ together imply that
$2x^2 + \phi'(y) = 2x^2 + 2y \Rightarrow \phi(y) = y^2 + c_1$ for some constant c_1. Hence the solution is
$f(x, y) = c$ or $x^3 + 2x^2y + y^2 + c_1 = c$.

Converting a first order non-exact DE to exact DE

Consider the following example:

Example

The first order DE $ydx - xdy = 0$ is clearly not exact. But observe that if we multiply both sides of this DE by $\frac{1}{y^2}$, the resulting ODE becomes $\frac{dx}{y} - \frac{x}{y^2}dy = 0$ which is exact!

Definition

It is sometimes possible that even though the original first order DE $M(x,y)dx + N(x,y)dy = 0$ is not exact, but we can multiply both sides of this DE by some function (say, $\mu(x,y)$) so that the resulting DE $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ becomes exact. Such a function/factor $\mu(x,y)$is known as an **integrating factor** for the original DE $M(x,y)dx + N(x,y)dy = 0$.

Remark: *It is possible that we LOSE or GAIN solutions while multiplying a ODE by an integrating factor.*

How to find an integrating factor?

We will now list down some rules for finding integrating factors, but before that, we need the following definition:

Definition

A function $f(x, y)$ is said to be **homogeneous** of **degree** n if $f(tx, ty) = t^n f(x, y)$ for all (x, y) and for all $t \in \mathbb{R}$.

Example

1) $f(x, y) = x^2 + y^2$ is homogeneous of degree 2.
2) $f(x, y) = tan^{-1}(\frac{y}{x})$ is homogeneous of degree 0.
3) $f(x, y) = \frac{x(x^2+y^2)}{y^2}$ is homogeneous of degree 1.
4) $f(x, y) = x^2 + xy + 1$ is NOT homogeneous.

How to find an integrating factor? contd...

Definition

A first order DE of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous** if both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.
NOTE: Here the word “homogeneousÂ¨ does not mean the same as it did for first order linear equation $a_1(x)y' + a_0(x)y = g(x)$ when $g(x) = 0$.

Some rules for finding an integrating factor: Consider the DE

$$M(x, y)dx + N(x, y)dy = 0. \qquad (*)$$

Rule 1: If $(*)$ is a homogeneous DE with $M(x, y)x + N(x, y)y \neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor for $(*)$.

How to find an integrating factor? contd...

Rule 2: If $M(x,y) = f_1(xy)y$ and $N(x,y) = f_2(xy)x$ and $Mx - Ny \neq 0$, where f_1 and f_2 are functions of the product xy, then $\frac{1}{Mx-Ny}$ is an integrating factor for $(*)$.

Rule 3: If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for $(*)$.

Rule 4: If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = F(y)$ (function of y-alone), then $e^{-\int F(y)dy}$ is an integrating factor for $(*)$.

Proof of Rule 3

Proof.

Let $f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$. To show: $\mu(x) := e^{\int f(x)dx}$ is an integrating factor. That is, to show $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$.

Since μ is a function of x alone, we have $\frac{\partial}{\partial y}(\mu M) = \mu \frac{\partial M}{\partial y}$. Also $\frac{\partial}{\partial x}(\mu N) = \mu'(x)N + \mu(x)\frac{\partial N}{\partial x}$.

So we must have:

$\mu(x)[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = \mu'(x)N$, or equivalently we must have,

$\frac{\mu'(x)}{\mu(x)} = f(x)$,

which is anyways true since $\mu(x) := e^{\int f(x)dx}$. □

The proof of Rule 4 is similar. The proof of Rule 2 is an exercise.

Another rule for finding an I.F.

- If a differential equation is in the special form

$$y(Ax^p y^q + Bx^r y^s)dx + x(Cx^p y^q + Dx^r y^s)dy = 0,$$

where A, B, C, D are constants, then an I.F. has the form $\mu(x, y) = x^a y^b$, where a and b are suitably chosen constants.

Solution by substitution

Often the first step of solving a differential equation consists of transforming it into another differential equation by means of a ***substitution****.*

For example, suppose we wish to transform the first order differential equation $\frac{dy}{dx} = f(x,y)$ *by the substitution* $y = g(x,u)$*, where* u *is regarded as a function of the variable* x*. If* g *possesses first partial derivatives, then the chain rule*

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx}$$

gives $\frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$*. The original differential equation* $\frac{dy}{dx} = f(x,y)$ *now becomes* $g_x(x,u) + g_u(x,u)\frac{du}{dx} = f(x, g(x,u))$*. This equation is of the form* $\frac{du}{dx} = F(x,u)$*, for some function* F*. If we can determine a solution* $u = \phi(x)$ *of this last equation, then a solution of the original differential equation will be* $y = g(x, \phi(x))$*.*

Use of substitution : Homogeneous equations

Recall: A first order differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be **homogeneous** if both M and N are homogeneous functions of the same degree.
Such equations can be solved by the substitution : $y = vx$.

Example

Solve $x^2ydx + (x^3 + y^3)dy = 0$.
Solution: The given differential equation can be rewritten as $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$.
Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Putting this in the above equation, we get $v + x\frac{dv}{dx} = \frac{v}{1+v^3}$. Or in other words, $(\frac{1+v^3}{v^4})dv = -\frac{dx}{x}$, which is now in separable variables form.

DE reducible to homogeneous DE

For solving differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

use the substitution

- $x = X + h$ and $y = Y + k$, if $\frac{a}{a'} \neq \frac{b}{b'}$, where h and k are constants to be determined.
- $z = ax + by$, if $\frac{a}{a'} = \frac{b}{b'}$.

Example

Solve $\frac{dy}{dx} = \frac{x+y-4}{3x+3y-5}$.
Observe that this DE is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $\frac{a}{a'} = \frac{b}{b'}$.
Use the substitution $z = x + y$. Then we have $\frac{dz}{dx} = 1 + \frac{dy}{dx}$. Putting these in the given DE, we get $\frac{dz}{dx} - 1 = \frac{z-4}{3z-5}$, or in other words, $\frac{3z-5}{4z-9}dz = dx$. This equation is now in separable variables form.

DE reducible to homogeneous DE, contd...

Example

Solve $\frac{dy}{dx} = \frac{x+y-4}{x-y-6}$.

Observe that this DE is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $1 = \frac{a}{a'} \neq \frac{b}{b'} = -1$.

Put $x = X + h$ and $y = Y + k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and

$$\frac{dY}{dX} = \frac{X + Y + (h + k - 4)}{X - Y + (h - k - 6)}. \qquad (*)$$

If h and k are such that $h + k - 4 = 0$ and $h - k - 6 = 0$, then $(*)$ becomes

$$\frac{dY}{dX} = \frac{X + Y}{X - Y}$$

which is a homogeneous DE. We can easily solve the system

$$h + k = 4$$

$$h - k = 6$$

of linear equations to detremine the constants h and k!

Reduction to separable variables form

A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C),$$

where A, B, C are real constants with $B \neq 0$ can always be reduced to a differential equation with separable variables by means of the substitution $u = Ax + By + C$. Observe that since $B \neq 0$, we get $\frac{u}{B} = \frac{A}{B}x + y + \frac{C}{B}$, or in other words, $y = \frac{u}{B} - \frac{A}{B}x - \frac{C}{B}$. This implies that $\frac{dy}{dx} = \frac{1}{B}(\frac{du}{dx}) - \frac{A}{B}$. Hence we have $\frac{1}{B}(\frac{du}{dx}) - \frac{A}{B} = f(u)$, that is, $\frac{du}{dx} = A + Bf(u)$. Or in other words, we have $\frac{du}{A+Bf(u)} = dx$, which is now in separable variables form.

Equations reducible to linear DE: Bernoullii's DE

Definition

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1}$$

where n is any real number, is called **Bernoullii's differential equation**.

Note that when $n = 0 \; or \; 1$, Bernoullii's DE is a linear DE.

<u>Method of solution:</u> Multiply by y^{-n} throughout the DE (1) to get

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x). \tag{2}$$

Use the substitution $z = y^{1-n}$. Then $\frac{dz}{dx} = (1-n)\frac{1}{y^n}\frac{dy}{dx}$. Substituting in equation (2), we get $\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$, which is a linear DE.

Example of Bernoullii's DE

Example

Solve the Bernoullis DE $\frac{dy}{dx} + y = xy^3$.
Multiplying the above equation throughout by y^{-3}, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x.$$

Putting $z = \frac{1}{y^2}$, we get $\frac{dz}{dx} - 2z = -2x$, which is a linear DE.
The integrating factor for this linear DE will be $= e^{-\int 2dx} = e^{-2x}$. Therefore, the solution is $z = e^{2x}[-2\int xe^{-2x}dx + c] = x + \frac{1}{2} + ce^{2x}$. Putting back $z = \frac{1}{y^2}$ in this, we get the final solution $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$.

Ricatti's DE

The differential equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

is known as **Ricatti's differential equation**. A Ricatti's equation can be solved by method of substitution, provided we know a paticular solution y_1 of the equation.
Putting $y = y_1 + u$ in the Ricatti's DE, we get

$$\frac{dy_1}{dx} + \frac{du}{dx} = P(x) + Q(x)[y_1 + u] + R(x)[y_1^2 + u^2 + 2uy_1].$$

But we know that y_1 is a particular solution of the given Ricatti's DE. So we have $\frac{dy_1}{dx} = P(x) + Q(x)y_1 + R(x)y_1^2$. Therefore the above equation reduces to

$$\frac{du}{dx} = Q(x)u + R(x)(u^2 + 2uy_1)$$

or, $\frac{du}{dx} - [Q(x) + 2y_1(x)R(x)]u = R(x)u^2$, which is Bernoulli's DE.

Orthogonal Trajectories

Suppose

$$\frac{dy}{dx} = f(x, y)$$

represents the DE of the family of curves. Then, the slope of any orthogonal trajectory is given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \quad \text{or} \quad -\frac{dx}{dy} = f(x, y),$$

which is the DE of the orthogonal trajectories.

Example: Consider the family of circles $x^2 + y^2 = c^2$. Differentiate w.r.t x to obtain $x + y\frac{dy}{dx} = 0$. The differential equation of the orthogonal trajectories is $x + y\left(-\frac{dx}{dy}\right) = 0$. Separating variable and integrating we obtain $y = c\,x$ as the equation of the orthogonal trajectories.