Ordinary Differential Equations

(MA102 Mathematics II)

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Method of Undetermined Coefficients

To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \qquad (1)$$

we must do two things:
(i) Find a complementary function y_c.
(ii) Find any particular solution y_p of the nonhomogeneous equation.
Then the general solution of (1) is $y = y_c + y_p$. We already know how to find y_c when the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants. We will now prescribe a method to find a particular solution y_p called **the method of undetermined coefficients**.
The underlying idea in this method is an educated guess, about the form of y_p, motivated by the kind of functions that comprise $g(x)$. This method is regretably limited to nonhomogeneous linear equations like (1) where

- the coefficients $a_i, i = 0, 1, \ldots, n$ are constants and
- $g(x)$ is a constant k, or a polynomial function, or an exponential function $e^{\alpha x}$, or sine or cosine function (like $sin\beta x, cos\beta x$), or finite sums and products of these functions.

Undetermined Coefficients contd ...

That is, $g(x)$ is a linear combination of functions of the type

$$k(constant), x^n, x^n e^{\alpha x}, x^n e^{\alpha x} cos\,\beta x, x^n e^{\alpha x} sin\,\beta x,$$

where n is a nonnegative integer and α and β are real numbers. The method of undetermined coefficients is not applicable to equations of form (1) when

$$g(x) = ln\ x, g(x) = \frac{1}{x}, g(x) = tan\ x, g(x) = sin^{-1} x,$$

ad so on. The set of functions that consists of constants, polynomials, exponentials, sines and cosines has the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials, sines and cosines. Since the lienar combination of derivatives $a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p$ must be identical to $g(x)$, it seems reasonable to assume that y_p has the same form as $g(x)$.

Examples

Example

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$.
It is easy to check that $y_c = c_1e^{-(2+\sqrt{6})x} + c_2e^{(-2+\sqrt{6})x}$. Now since the function $g(x)$ is a quadratic polynomial, let us assume that a particular solution y_p is also of the same form. So take

$$y_p = Ax^2 + Bx + C.$$

Then we get $y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$. We want $y_p'' + 4y_p' - 2y_p$ to be equal to $g(x) = 2x^2 - 3x + 6$. So equating $2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$ with $2x^2 - 3x + 6$, we can get the values of the coefficients A, B and C. Here we get $A = -1, B = -\frac{5}{2}, C = -9$. Thus a particular solution is $y_p = -x^2 - \frac{5}{2}x - 9$ and the general solution is

$$y = y_c + y_p = c_1e^{-(2+\sqrt{6})x} + c_2e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

Examples contd ...

Example

Solve $y'' - y' + y = 2sin\ 3x$.
A natural first guess for a particular solution would be $Asin\ 3x$. But since successive differentiations of $sin\ 3x$ produce $sin\ 3x$ as well as $cos\ 3x$, therefore we are prompted instead to assume a particular solution that includes both $sin\ 3x$ and $cos\ 3x$. So we take

$$y_p = Acos\ 3x + Bsin\ 3x.$$

And then we proceed similarly as in the previous example to compute the values of A and B.

Example: Forming y_p by superposition

Example

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.
In this example, the presence of $4x - 5$ in $g(x)$ suggests that the particular solution must include a linear polynomial. Furthermore, the presence of $6xe^{2x}$ in $g(x)$ and the fact that derivative of the product xe^{2x} produces $2xe^{2x}$ and e^{2x}, suggest that the particular solution must include both xe^{2x} and e^{2x}. In other words, g is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = polynomial + exponential.$$

Correspondingly, the superposition principle for nonhomogeneous equations suggest that we seek a particular solution

$$y_p = y_{p_1} + y_{p_2}$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + De^{2x}$. So take $y_p = Ax + B + Cxe^{2x} + De^{2x}$ and solve the DE as in the earlier example.

A glitch in the method

Example

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.
Here $g(x) = 8e^x$ and differentiation of e^x produces no new functions. Thus it seems reasonable to assume a particular solution of the form $y_p = Ae^x$. But substituting $y_p = Ae^x$ in the DE yields $0 = 8e^x$, which is absurd. So clearly we had made a wrong guess for y_p.
The difficulty here is apparent upon examining the complementary function $y_c = c_1e^x + c_2e^{4x}$. Observe that our assumption Ae^x is already present in y_c in the form c_1e^x. This means that e^x is a solution of the associated homogeneous DE, and a constant multiple Ae^x when substituted into the DE necessarily produces 0.
What then should be the form of y_p? Let us see whether we can find a particular solution of the form $y_p = Axe^x$ or not! Substituting $y_p = Axe^x$ in the given DE, we get $y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x$. This implies that $A = -\frac{8}{3}$. Hence a particular solution of the given DE is $y_p = -\frac{8}{3}xe^x$.

This example suggests us to consider 2 cases.

Case I

Recall the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \qquad (1)$$

Case I: No function in the assumed particular solution is a solution of the associated homogeneous DE.
In the table that follows, we illustrate some specific examples of $g(x)$ in (1) along with the corresponding form of the particular solution. We are, of course, taking for granted that no function in the assumed particular solution y_p is duplicated by a function in the complementary function y_c.

Form Rule for Case I: The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$.

Table for Case I

Trial particular solutions

$g(x)$	$Form\ of\ y_p$
$k(any\ constant)$	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+D
$sin\ \beta x$	$Acos\ \beta x+Bsin\ \beta x$
$cos\ \beta x$	$Acos\ \beta x+Bsin\ \beta x$
$e^{\alpha x}$	$Ae^{\alpha x}$
$(9x-2)e^{\alpha x}$	$(Ax+B)e^{\alpha x}$
$x^2e^{\alpha x}$	$(Ax^2+Bx+C)e^{\alpha x}$
$e^{\alpha x}sin\ \beta x$	$Ae^{\alpha x}cos\ \beta x+Be^{\alpha x}sin\ \beta x$
$5x^2sin\ \beta x$	$(Ax^2+Bx+C)cos\ \beta x+(Ex^2+Fx+G)sin\ \beta x$
$xe^{\alpha x}cos\ \beta x$	$(Ax+B)e^{\alpha x}cos\ \beta x+(Cx+D)e^{\alpha x}sin\ \beta x$

Example for Case I

Example

Find a particular solution of $y'' - 9y' + 14y = 3x^2 - 5sin\ 2x + 7xe^{6x}$.
Solution: Corresponding to $3x^2$ we assume $y_{p_1} = Ax^2 + Bx + C$.
Corresponding to $-5sin\ 2x$ we assume $y_{p_2} = Ecos\ 2x + F sin\ 2x$.
Corresponding to $7xe^{6x}$ we assume $y_{p_3} = (Gx + H)e^{6x}$.
The assumption for the particular solution for the given DE is then

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + Ecos\ 2x + F sin\ 2x + (Gx + H)e^{6x}.$$

Observe that no term in this assumed y_p duplicates any term in $y_c = c_1e^{2x} + c_2e^{7x}$.

Case II

Case II: A function in the assumed particular solution is also a solution of the associated homogeneous DE.

Example

Problem: Find a particular solution of $y'' - 2y' + y = e^x$.
The complementary function is $y_c = c_1 e^x + c_2 x e^x$. So the assumption $y_p = Ae^x$ will fail since it is apparent from y_c that e^x is a solution of the associated homogeneous equation $y'' - 2y' + y = 0$. Moreover, we will not be able to find a particular solution of the form $y_p = Axe^x$ since the term xe^x is also duplicated in y_c. We next try

$$y_p = Ax^2 e^x.$$

Substituting into the given DE yields

$$2Ae^x = e^x \text{ and so } A = \frac{1}{2}.$$

Thus a particular solution is $y_p = \frac{1}{2} x^2 e^x$.

Case II contd ...

Suppose again that $g(x)$ consists of m terms of the kind given in the previous table, and suppose further that the usual assumption for a particular solution is

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m},$$

where the y_{p_i}, $i = 1, 2, \cdots, m$ are the trial particular solution forms corresponding to these terms. Under the circumstances described in Case II, we can make up the following general rule:

Multiplication Rule for case II: If any y_{p_i} contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^s, where s is the smallest positive integer that eliminates that duplication.

Using the Multiplication Rule, Case II

Example

Problem: Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.
The complementary function is $y_c = c_1 e^{3x} + c_2 x e^{3x}$. And so, the usual assumption (based on the table) for a particular solution would be

$$y_p = Ax^2 + Bx + C + Ee^{3x}$$

where $y_{p_1} = Ax^2 + Bx + C$ and $y_{p_2} = Ee^{3x}$. Inspection of these functions show that the one term in y_{p_2} is duplicated in y_c. If we multiply y_{p_2} by x, we note that the term xe^{3x} is still a part of y_c. But multiplying y_{p_2} by x^2 eliminates all duplications. Thus the operative form of a particular solution is

$$y_p = Ax^2 + Bx + C + Ex^2 e^{3x}.$$

A comprehensive list

Form of y_p:

- $g(x) = p_n(x) = a_n x^n + \cdots + a_1 x + a_0$,
  $y_p(x) = x^s P_n(x) = x^s\{A_n x^n + \ldots + A_1 x + A_0\}$
- $g(x) = ae^{\alpha x}$, $y_p(x) = x^s A e^{\alpha x}$
- $g(x) = a\cos\beta x + b\sin\beta x$,
  $y_p(x) = x^s\{A\cos\beta x + B\sin\beta x\}$
- $g(x) = p_n(x)e^{\alpha x}$, $y_p(x) = x^s P_n(x)e^{\alpha x}$
- $g(x) = p_n(x)\cos\beta x + q_m(x)\sin\beta x$,
  $y_p(x) = x^s\{P_N(x)\cos\beta x + Q_N(x)\sin\beta x\}$,
  where $q_m(x) = b_m x^m + \cdots + b_1 x + b_0$,
  $Q_N(x) = B_N x^N + \cdots + B_1 x + B_0$ and $N = \max(n, m)$

A comprehensive list contd ...

- $g(x) = ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$,
  $y_p(x) = x^s\{Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x\}$
- $g(x) = p_n(x)e^{\alpha x}\cos + q_m(x)e^{\alpha x}\sin\beta x$,
  $y_p(x) = x^s e^{\alpha x}\{P_N(x)\cos\beta x + Q_N(x)\sin\beta x\}$, where
  $N = \max(n, m)$.

Note:

1. The nonnegative integer s is chosen to be the smallest integer so that no term in y_p is a solution to $L(y) = 0$.

2. $P_n(x)$ must include all its terms even if $p_n(x)$ has some terms that are zero.