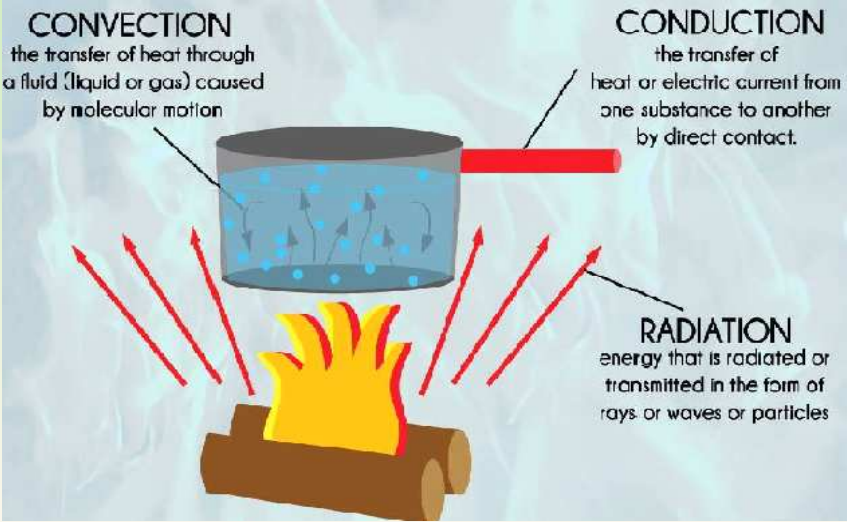
MA542 January-May2022, Differential Equations 1D heat conduction equation

Lecture 39

19/04/2022

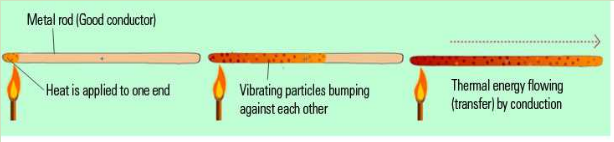
Heat Conduction Problem



Heat conduction in a thin rod

One important problem is the heat conduction in a thin metallic rod of finite length.

This transient problem represents a class of problem known as diffusion problems.



Just like one-dimensional wave equation, separation of variables can be appropriately applied.

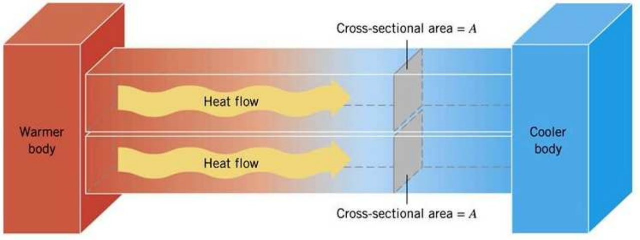
The IBVP under consideration for heat conduction in a thin metallic rod of length L consists of:

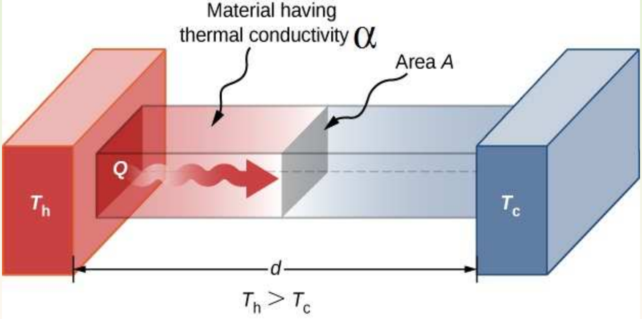
The governing equation

$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0, \tag{1}$$

where α is the thermal diffusivity.

Heat conduction in a thin rod (Contd.)





Heat conduction in a thin rod (Contd.): Case I
Homogeneous Dirichlet boundary condition

The boundary conditions for all $t > 0$ due to maintaining the temperature at both ends at zero degrees:

$$u(0,t) = 0, \tag{2a}$$
$$u(L,t) = 0. \tag{2b}$$

The initial condition for $0 \leq x \leq L$, i.e., initial temperature distribution:

$$u(x,0) = \phi(x). \tag{3}$$

The assumptions are:

- The rod is very thin so that it can be considered as one-dimensional.
- The curved surface is insulated.
- There is no external source of heat.

Using the separation of variable technique by assuming a solution of the form

$$u(x,t) = X(x)T(t).$$

Heat conduction in a thin rod (Contd.)

The governing equation is converted to the following ODEs:

$$\begin{aligned} X'' - kX &= 0, \\ T' - k\alpha T &= 0. \end{aligned}$$

where k is a separation constant.

Observe that the boundary conditions for this IBVP are the same as those in the one-dimensional wave equation.

We can immediately assume that a feasible nontrivial solution exists only for the negative values of k.

Above equations $\Rightarrow$

$$X'' + \lambda^2 X = 0, \tag{4}$$

$$T' + \lambda^2 \alpha T = 0. \tag{5}$$

Heat conduction in a thin rod (Contd.)

The solutions:

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x), \tag{6}$$

$$T(t) = Ce^{-\alpha\lambda^2 t}. \tag{7}$$

Solution for $u(x,t)$:

$$u(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]Ce^{-\alpha\lambda^2 t}. \tag{8}$$

Using the boundary conditions (2a) and (2b)

$B = 0$ and $\lambda_n = \dfrac{n\pi}{L},\ n = 1, 2, 3, \ldots.$

Solution corresponding to each n:

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\ e^{-\alpha\frac{n^2\pi^2}{L^2}\ t}.$$

Heat conduction in a thin rod

The general solution:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\ e^{-\alpha \frac{n^2\pi^2}{L^2}\, t} \tag{9}$$

where A_n is obtained (refer to the solution of the one-dimensional wave equation) by using the initial condition (3):

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\ dx,\ \ n = 1, 2, 3, \ldots \tag{10}$$

Observe that what we have solved here is a homogenous equation

subject to homogenous (zero) conditions.

Ideally the boundary conditions are very likely to be non-homogenous for diffusion problems.

Example

Example

A thin metal bar of length π is placed in boiling water (temperature 100^0 degrees). After reaching 100^0 throughout, the bar is removed from the boiling water and immediately, at time $t = 0$, the ends are immersed and kept in a medium with constant freezing temperature 0^0 degrees. Taking the thermal diffusivity of the metal to be one, find the temperature $u(x,t)$ for $t > 0$.

Solution: The IBVP will be

$$u_t = u_{xx}, \;\; u(0,t) = 0 = u(\pi,t), \;\; u(x,0) = 100$$

We know for a rod of length L

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \exp\left(-\alpha n^2\pi^2/L^2\right)t \quad \text{with}$$

$$A_n = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}dx.$$

Examples

We evaluate A_n as

$$A_n = \frac{2}{\pi}\int_0^{\pi} 100 \sin nx \; dx = \frac{200}{n\pi}(1 - \cos n\pi)$$

The solution:

$$u(x,t) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x \quad \exp\{-(2n-1)^2 t\}.$$

Heat conduction in a thin rod: Case II
Non-homogeneous Dirichlet boundary conditions

With nonhomogeneous boundary conditions

Consider the heat conduction in a thin metal rod of length L with insulated sides

with the ends $x = 0$ and $x = L$ kept at temperatures $u = {u_0}^0$C and $u = {u_L}^0$C, respectively, for all time $t > 0$.

Suppose that the temperature distribution at $t = 0$ is $u(x, 0) = \phi(x),\ 0 < x < L$.

To determine the temperature distribution in the rod at some subsequent time $t > 0$.

Heat conduction in a thin rod

The IBVP under consideration consists of

The governing equation:

$$u_t = \alpha u_{xx}, \tag{11}$$

where α is the coefficient of thermal diffusivity for the specific metal.

The boundary conditions for all $t > 0$:

$$u(0,t) = u_0, \tag{12a}$$

$$u(L,t) = u_L. \tag{12b}$$

The initial conditions for $0 \leq x \leq L$:

$$u(x,0) = \phi(x). \tag{13}$$

Heat conduction in a thin rod

Due to the nonhomogeneous boundary conditions (12),

the direct application of the method of separation of variables will not work.

We intend to convert the given IBVP into two problems:

one IBVP resembling the problem we have solved previously

and the other one a BVP taking care of the nonhomogeneous terms.

Seek a solution in the form:

$$u(x,t) = v(x,t) + h(x), \tag{14}$$

where $h(x)$ is an unknown user defined function of x alone to take care of the non-homogeneous boundary conditions such that

$$h(0) = u_0 \ \ \& \ \ h(L) = u_L.$$

Heat conduction in a thin rod

Using (14) in $u_t - \alpha u_{xx} = 0$, we have

$$v_t = \alpha[v_{xx} + h''(x)] \tag{15}$$

subject to

$$v(0,t) + h(0) = u_0, \tag{16a}$$
$$v(L,t) + h(L) = u_L, \tag{16b}$$

and

$$v(x,0) + h(x) = \phi(x). \tag{17}$$

Now equations (15)-(17) can be conveniently split into two problems as follows:

Heat conduction in a thin rod

Problem I: BVP

$$\begin{aligned} \alpha h''(x) &= 0, \\ h(0) &= u_0,\ h(L) = u_L. \end{aligned}$$

Problem II: IBVP

$$\begin{aligned} &v_t = \alpha v_{xx}, \\ &v(0,t) = 0 = v(L,t), \\ &v(x,0) = \phi(x) - h(x). \end{aligned}$$

The solution of Problem II is known to us from **Case I**, which is

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\ e^{-\alpha\frac{n^2\pi^2}{L^2}\ t}, \tag{18}$$

with

$$A_n = \frac{2}{L}\int_0^L (\phi(x) - h(x))\sin\frac{n\pi x}{L}\ dx,\ n = 1, 2, 3, \ldots \tag{19}$$

Heat conduction in a thin rod

The solution for Problem I can be easily found in two steps by integrating $h''(x)$ and $h'(x)$ in succession:

$$h(x) = Ax + B.$$

Upon using the boundary conditions, we get

$B = u_0$ and $A = u_L - u_0$

Hence

$$h(x) = \frac{u_L - u_0}{L}\, x + u_0. \tag{20}$$

Therefore, the solution $u(x,t)$ for given IBVP is given by

the sum of (18) and (20).

Heat conduction problem with flux and radiation conditions

The previous two problems that we considered represented situations where we have fixed the temperatures at the ends of the rod,

and are examples of Dirichlet or fixed endpoint conditions.

There are two other types of problems concerning the heat conduction in a thin rod.

These types of problems contain other types of boundary conditions:

the first one is a Neumann or flux condition,

the second one is a Robin or radiation condition.

Heat conduction problem with flux and radiation conditions

An insulation-type flux condition at, say, $x = 0$ is a condition on the derivative of the form $u_x(0, t) = 0, t > 0$;

because the flux is proportional to the temperature gradient (Fourier's heat flow law states flux $= -\alpha u_x$, where α is the conductivity).

Heat flux is defined as the rate of heat transfer per unit cross-sectional area.

The insulation condition requires that

no heat flow across the boundary $x = 0$.

A radiation condition, on the other hand, is a specification of how heat radiates

from the end of the rod, say at $x = 0$, into the environment, or how the end absorbs heat from its environment.

Linear, homogeneous radiation condition takes the form

$-\alpha u_x(0, t) + bu(0, t) = 0, t > 0$, where b is a constant.

Heat conduction problem with flux and radiation conditions

If $b > 0$, then the flux is negative

which means that heat is flowing from the rod into its surroundings (radiation)

if $b < 0$, then the flux is positive,

and heat is flowing into the rod (absorption).

A typical problem in heat conduction may have

a combination of Dirichlet, insulation and radiation boundary conditions.

Heat conduction problem in a thin rod

Consider following heat conduction equation in a rod $(0, L)$

$$u_t = \alpha u_{xx}, \;\; 0 < x < L, \;\; t > 0. \tag{21}$$

The initial condition for $0 \leq x \leq L$:

$$u(x, 0) = f(x). \tag{22}$$

In practice, temperature $u(x,t)$ satisfies certain boundary conditions such as:

(a) Dirichlet Condition: $u(0,t) = u_0, \;\; u(L,t) = u_L, \; t > 0$

(b) Neumann Condition: $u_x(0,t) = \alpha_1, \;\; u_x(L,t) = \beta_1, \; t > 0,$

(c) Robin Condition: $u_x(0,t) + a_0 u(0,t) = \alpha_2, \;\; u_x(L,t) + a_L u(L,t) = \beta_2.$

Physically, u_x denotes heat flux. So, $u_x(0,t) = 0$ means

the left end of the rod is insulated, i.e., heat transfer through that point is zero.