1. Recall definition of a domain A in $\mathbb{R}$. Quick mention of *limit of g at c*, for a function $g : A \to \mathbb{R}$ at a point $c \in A$. (Deliberately exclude $x = c$, anticipating definition of derivative).

2. Analogous to limit of a sequence and definition of continuity. Comparable results are true: uniqueness of limit if it exists and the rules: power, sum, difference, product, quotient (be delicate), chain/composition. Squeeze theorem. Write down proofs by yourself.

3. Galileo's law of motion states that an object/body/particle without any external force/disturbance continues its state of rest *or* uniform velocity (constant speed in a straight line).
   Challenges faced by Newton:

   (a) To give an *absolutely accurate* definition of *instantaneous* velocity from the crude calculations of *average* velocity.

   (b) Having answered (a), to quantify amount of *instantaneous* change in velocity.

4. Certainly, you have seen the definition of instantaneous velocity for a particle whose position has been co–ordinated by real numbers. It is the 'eventual value' of average velocity quotient $\frac{f(c+h)-f(c)}{h}$ as the time window h 'becomes'/'approaches' zero. The definition is accurate. Indeed, using this and an analogous definition for *instantaneous* acceleration, Newton calculated that moon is falling towards earth at a rate $\frac{1}{60^2}$ that of a stone thrown up. Knowing that the distance between moon and earth is 60 times the radius of earth – he hypothesised the Universal Law of Gravitation.

5. Latter day mathematicians have made these notions rigorous. Stripped of any and all physical interpretation, consider a function $f : A \to \mathbb{R}$ for a domain A in $\mathbb{R}$. For a $c \in A$, we say that
   $$f \text{ is } \textit{differentiable} \text{ at } c \in A \text{ if } \quad \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ at } c \text{ exists.}$$
   Note that $g(h) = \frac{f(c+h)-f(c)}{h}$ is defined on a subset of $\mathbb{R}$ *not* containing 0 and we are seeking $\lim\limits_{h\to 0} g(h)$.

6. Define *differentiable* to mean differentiable at all points of the domain.

7. Notation for derivative.

8. Proposition: Differentiable at c implies continuous at c.

9. Rules: Power, Sum, Difference, Product, Quotient, Chain. Proof of chain rule is delicate. See BaSh.

10. The inverse function theorem in the world of continuous functions assures a continuous inverse for a bijective continuous function. Bijectivity is necessary if you want to invert a function. And if a function is continuous, by the inverse function theorem, the inverse function is continuous. (all this is for functions between intervals).
    What should one do to get a differentiable inverse? Certainly we need a bijective function. And if the function is differentiable, it is continuous and by our previous observation, the inverse function is continuous. But is the inverse function differentiable? The function $x \mapsto x^{\frac{1}{3}}$ from $(-1, 1)$ to $(-1, 1)$ is not differentiable (at zero) but has a differentiable inverse, viz., $x \mapsto x^3$. Likewise, the function $x \mapsto x^3$ is differentiable and has a local differentiable inverse at all points of its domain except zero.
    These two examples illustrate that for bijective functions, differentiability is neither necessary nor sufficient to get a differentiable inverse. This is explained by the observation (in examples, at least) $f'(x_n) \to 0$ as $x_n \to c$ causes $|f^{-1\prime}(y_n)| \to \infty$ as $y_n = f(x_n) \to f(c)$.

11. Infinitesimal Inverse Function Theorem (in the world of differentiable functions): Let $f : I \to J$ be a bijective continuous function between intervals I and J, with $g : J \to I$ as the inverse. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $f(c)$ and $g'(f(c)) = \frac{1}{f'(c)}$.

12. Local Inverse Function Theorem: Let $f : I \to J$ be a bijective continuous function between intervals I and J, with $g : J \to I$ as the inverse. Suppose that for a $c \in I$, there is a real $\epsilon > 0$ such that f is differentiable and f' is continuous on $(c - \epsilon, c + \epsilon)$ with $f'(c) \neq 0$. Then, there is a real $\delta > 0$ such that at every $x \in (c - \delta, c + \delta) \cap I$, g is differentiable on $f((c - \delta, c + \delta) \cap I)$ and $g'(f(x)) = \frac{1}{f'(x)}$.

13. Global Inverse Function Theorem: Let $f : I \to J$ be a bijective differentiable function between intervals I and J, with $g : J \to I$ as the inverse. If f is differentiable and $f'(x) \neq 0$ for every $x \in I$, then g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$ for every $x \in I$.

14. Discussion topics:

    (a) To justify the applicability and usefulness of Infinitesimal Inverse Function Theorem, produce examples for functions defined on intervals where the Local Inverse Function Theorem cannot be applied, but the Infinitesimal version can be applied. Hint: Can you find a function $f : [-1, 1] \to [-1, 1]$ such that f is bijective, continuous, f is differentiable at 0, $f'(0) \neq 0$, AND there exist a sequence $\alpha_n \to 0$ such that f is not differentiable at every α_n?

    (b) Challenge: Give an example for a function $f : \mathbb{R} \to \mathbb{R}$ such that f is bijective, f is differentiable at all points of a pre-specified subset $S \subset \mathbb{R}$ and not differentiable at points of S^c.

    (c) Another one: Can you find an example of a bijective function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable nowhere but the inverse is differentiable (everywhere).

1. Definitions for *relative minimum, maximum, extremum* for a function defined on an interval or even a domain. Note the use of the words *minimum/maximum* instead of *infimum/supremum*. The word *relative* contrasts these definitions with *absolute* used earlier.

2. Examples of existence and non–existence.

3. Derivative vanishes at interior points of relative extrema.
On an interval I, let $f : I \to \mathbb{R}$ be a function. Let c be an interior point of I, f have a relative extremum at c and let f be differentiable at c. Then $f'(c) = 0$.
Proof: Assume $f'(c) = \ell \neq 0$. Then there exists a real $\delta > 0$ such that if $0 < |h| < \delta$, then $\ell - \frac{1}{2}\ell < \frac{f(c+h)-f(c)}{h} < \ell + \frac{1}{2}\ell$. Consider:

   (a) Assume c is a point of relative maximum of f.

      i. If $\ell > 0$, using the above inequality, we have for all $0 < h < \delta$ that $f(c + h) - f(c) > \frac{1}{2}\ell \cdot h > 0$ which contradicts the assumption that c is a point of relative maximum.

      ii. If $\ell < 0$, using the above inequality, we have for all $-\delta < h < 0$ that $f(c + h) - f(c) > \frac{3}{2}\ell \cdot h$ which contradicts the assumption that c is a point of relative maximum.

   (b) The case of c being a point of relative minimum of f is similar.

   Remark: To make sense of $f(c + h)$, δ has to be sufficiently small. Since we are using both $h > 0$ and $h < 0$, c should be an interior point.

4. The proof above tells us that if f is differentiable (even just) at c and $f'(c) > 0$, then there exists a real $\delta > 0$ such that for all $0 < h < \delta$, $f(c + h) - f(c) > 0$ and $f(c - h) - f(c) < 0$. What this does not tell us is that $f(x) - f(y) > 0$ for all $x < y \in (c - \delta, c + \delta)$ – i.e., f is locally increasing. Indeed this is false as the example of $x \mapsto x + x^2 \sin(1/x)$ with $0 \mapsto 0$ illustrates. Upshot: From the infinitesimal data $f'(c) > 0$, we cannot conclude f is locally increasing.

5. Critical points of a function on a domain are defined as the collection which includes end points of the domain, those points where derivative fails to exist or those where the derivative exists and equals zero. If a function has a relative extremum at a point, then that point is a critical point.

6. Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists a (at least one) point $\xi \in (a, b)$ such that $f'(\xi) = 0$. (Here $a < b$)
Proof: If $\sup f = \inf f = f(a) = f(b)$, then f is a constant and any element in (a, b) may be used for ξ. Otherwise, an interior point ξ of the interval is a point of absolute extremum and hence relative extremum. By previous theorem, $f'(\xi) = 0$.

7. Mean Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a (at least one) point $\xi \in (a, b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. (Here $a < b$)
Proof: Use Rolle's Theorem on $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$.
Example of Speeding Fine.

8. First Derivative Test. For a real number $\delta > 0$, a function $f : (c - \delta, c + \delta) \to \mathbb{R}$ has the following properties:

   (a) f is continuous

   (b) f is differentiable for all $x \in (c - \delta, c)$ and for all $x \in (c, c + \delta)$.

   (c) For all $x \in (c - \delta, c)$, $f'(x) \geq 0$

   (d) For all $x \in (c, c + \delta)$. $f'(x) \leq 0$.

   Then, f has a relative maximum at c. Analogous statement for relative minimum.

   Proof: ?