1. Lengths: Definition of a unit length as a standard. Lengths of straight line segments which are natural numbers, fractions and real numbers.

2. Areas: Definition of a unit square as standard.

3. Area of a rectangle whose sides are of lengths which are natural numbers/non–negative integers. Extension to rectangles having sides of lengths which are non–negative fractions. How about if sides are non–negative reals?

4. Area of a parallelogram. Area of a triangle. Area of a polygon.

5. Remark that with the definitions we have, the most general curves for which we can *ask* about their lengths are actually only straight line segments. Similarly the most general regions for which we can *ask* about its area are polygons. Of course, minor variants like lengths of piece–wise straight line segments and a collection of polygons any two of which intersect either in an empty set or in a point are allowed. It is *inappropriate* to ask about circumference of a circle or an ellipse or area of a circular or an elliptical region, for instance.

6. Despite the remark, ancients considered the problem of lengths of more general curves and areas of more general planar regions. For calculating lengths of "curved" curves, their method was to approximate the curve by a piece–wise straight line segment and take "finer and finer" approximations. The method for calculating areas of regions with "curved" boundaries is to approximate the region by a collection of triangles or rectangles and take "finer and finer" approximations.

7. What is the circumference of a circle of radius R? What is its area?

8. Proposition: There exists a universal constant denoted by π such that for any circle $\frac{\text{circumference}}{\text{diameter}} = \frac{\text{area}}{(\text{radius})^2} = \pi$.
Proof: (Using Trigonometry) Let $\mathcal{C}$ and $\mathcal{A}$ denote the circumference and area of a circle of radius R. Let p_n, P_n, a_n, A_n denote respectively the perimeters and areas of regular n-gons ($n \geq 3$) circumscribed by and circumscribing the given circle. We then have
$$p_n < \mathcal{C} < P_n \quad \& \quad a_n < \mathcal{A} < A_n \qquad \text{for every natural } n \geq 3.$$
Using trigonometry, one verifies
$$p_n = 2R \cdot 180 \cdot \frac{\sin\left(\frac{180^\circ}{n}\right)}{\frac{180}{n}}, \quad P_n = 2R \cdot 180 \cdot \frac{\tan\left(\frac{180^\circ}{n}\right)}{\frac{180}{n}}, \qquad a_n = R^2 \cdot 180 \cdot \frac{\sin\left(\frac{360^\circ}{n}\right)}{\frac{360^\circ}{n}}, \quad A_n = R^2 \cdot 180 \cdot \frac{\tan\left(\frac{180^\circ}{n}\right)}{\frac{180}{n}}$$
Can you prove that there is a real number α such that $\lim\limits_{\theta^\circ \to 0} \frac{\sin(\theta^\circ)}{\theta^\circ} = \lim\limits_{\theta^\circ \to 0} \frac{\tan(\theta^\circ)}{\theta^\circ} = \alpha$? ...(*). Define π to be that real number which is 180α. Then, one has $\lim p_n = \lim P_n = \mathcal{C} = 2\pi R$ and $\lim a_n = \lim A_n = \mathcal{A} = \pi R^2$. This completes the proof.
Note that your proof of (*) should *not* use $\lim \frac{\sin \xi}{\xi} = 1$, when ξ is measured in radians. This is owing to the fact that radian measure presupposes establishment of π and length of circular arc.

9. Can one find the circumference and area of a circle without using trigonometry?

10. Lemma: If s_n and S_n are the lengths of sides of regular n–gons circumscribed by and circumscribing a circle of radius R, then
$$\frac{s_{2n}}{R} = \sqrt{2 - 2\sqrt{1 - \left(\frac{s_n}{2R}\right)^2}} \quad \& \quad \frac{S_{2n}}{R} = \frac{4}{\frac{S_n}{R}} \left(\sqrt{1 + \left(\frac{S_n}{2R}\right)^2} - 1\right)$$
Proof: Elementary Euclidean Geometry.

11. Antiphon found $\lim p_{2^k}$, Bryson found $\lim P_{2^k}$ and Archimedes found $\lim A_{2^k}$ using the above lemma as follows. One can verify that $s_4 = \sqrt{2}R$ and $S_4 = 2R$. Using the lemma, the sequences (s_{2^k}) and (S_{2^k}) are well defined. As an exercise prove that $(2^k s_{2^k}), (2^k S_{2^k})$ and $(2^k \frac{1}{2} R S_{2^k})$ converge to $2\pi R, 2\pi R$ and πR^2 for some real number π.

12. Corollary: Define $\alpha_2 = \sqrt{2}$ and $\alpha_{k+1} = \sqrt{2 - 2\sqrt{1 - \left(\frac{\alpha_k}{2}\right)^2}}$ for each $k \geq 2$.
Similarly, $\beta_2 = 2$ and $\beta_{k+1} = \frac{4}{\beta_k}\left(\sqrt{1 + \left(\frac{\beta_k}{2}\right)^2} - 1\right)$ for each $k \geq 2$.
$2^{k-1}\alpha_k$ and $2^{k-1}\beta_k$ are monotonic sequences converging to π.

1. Ancient Problem of Areas: Given a region in a plane, find its area. In this form, the problem is ill–posed. Despite this, like the example of circle, mathematicians of antiquity found area of elliptical regions, area of parabolic regions etc.

2. Issues with their methods: What is a definition for *area* of a region? If a sequence of polygons approximate a region, over what kinds of such polygonal approximations should one take the limit to get the area of the region? Does every such limit yield the same answer?

3. The biggest challenge was to get the *exact* answer. Exact answers could be found in very few examples. There was nothing anyone could do for almost two thousand years.

4. We shall consider the problem of *area under a function* $f : [a, b] \to \mathbb{R}$. By this we mean the area of the region whose boundary is determined by the curves $x = a$, $x = b$ and $y = f(x)$. Question: Is this restricted problem sufficient to solve the Ancient Problem of Areas?

5. Given an interval $[a, b]$, define a *partition* for that interval, *subinterval*, *norm* of a partition, *tag* for a subinterval and *tagged* partition.

6. The Riemann sum of a function defined on a compact interval w.r.t. a tagged partition is an approximation to the yet-to-be-defined area under that function. One should find limits of Riemann sums over a sequence of tagged partitions whose norm goes to zero. If all such limits yield the same answer, we say that *area under the function* exists/well–defined and take that common answer to be the value of the area.

7. We say that $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* or simply *integrable* if any of the following equivalent conditions hold:
   <u>Sequential Criterion</u> There exists a real number L such that if $\dot{\mathcal{P}}_n$ is a sequence of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$, then $\lim S(f, \dot{\mathcal{P}}_n) = L$.
   <u>Riemann's Criterion</u> There exists a real number L such that for any real $\epsilon > 0$, there exists a real $\delta > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ with $||\dot{\mathcal{P}}|| < \delta$, we have $|S(f, \dot{\mathcal{P}}) - L| < \epsilon$.
   <u>Cauchy's Criterion</u> For every real $\epsilon > 0$, there exists a real $\eta > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions of $[a, b]$ with $||\dot{\mathcal{P}}||, ||\dot{\mathcal{Q}}|| < \eta$, we should have $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \epsilon$.

8. If any of the above criteria hold, we say that f is integrable (over $[a, b]$) and its (Riemann) integral is L. We write $\int_a^b f(x)dx = L$. This notation is unfortunate as it anticipates fundamental theorem of calculus.

9. Uniqueness of limit.

10. Proof: RC implies SC, CC are easy. For SC implies RC, do rather OPP(RC) implies OPP(SC). For CC implies RC, see BaSh.

11. Examples of integrability of constant, identity and square functions on compact intervals.

12. Rules: Sum, Difference and Constant Multiple Rules of Riemann Integrability. Dominance/Comparison Rule. All proofs are easy with Sequential Criterion.

13. Integrability of restrictions, Summation over restrictions.

14. Theorems: integrable implies bounded, continuous implies integrable and monotonic implies integrable. How about their converses?

15. Definition of integral for $a = b$ and $b < a$. Interpretation for negative areas.