1. Natural numbers or Counting numbers : $\mathbb{N} = \{1, 2, 3, \ldots\}$.
   (a) Addition is closed, associative and commutative.
   (b) Multiplication is closed, associative and commutative. 1 is the unique multiplicative identity.
   (c) Multiplication distributes over addition.
   (d) There is no additive identity. We cannot talk of additive inverses.
   (e) There are no multiplicative inverses except for 1.
   (f) Principle of mathematical induction is valid, *viz.*, Assume $P(n)$ is a well-defined statement for each natural number n which is either true or false. If (i) $P(1)$ is true and (ii) For each natural k, $P(k)$ true implies $P(k+1)$ true, then: $P(n)$ is true for each natural n.
   (g) There is an order relation $1 < 2 < 3 < \cdots$
   (h) {Well–ordering principle} Every non-empty subset of naturals has a least element.
   (i) ...... many other derived/inferred properties.
   (j) Subtraction?

2. Integers : $\mathbb{Z} = \{1, 2, 3, \ldots, 0, -1, -2, -3, \ldots\}$.
   (a) Addition is closed, associative and commutative. 0 is the unique additive identity.
   (b) Multiplication is closed, associative and commutative. 1 is the unique multiplicative identity.
   (c) Multiplication distributes over addition.
   (d) Every integer has a unique additive inverse.
   (e) There are no multiplicative inverses except for ± 1.
   (f) There is an order relation $\cdots -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$
   (g) Subtraction is a closed operation.
   (h) ...... many other derived/inferred properties
   (i) Principle of mathematical induction?
   (j) Does every non-empty subset of integers have a least element?
   (k) Division?

3. Rationals : $\mathbb{Q} = \{1, 2, 3, \ldots, 0, -1, -2, -3, \ldots \frac{1}{2}, \frac{3}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2}, \ldots, \frac{1}{3}, \frac{2}{3}, \ldots, -\frac{1}{3}, -\frac{2}{3}, \ldots, \}$.
   (a) Addition is closed, associative and commutative. 0 is the unique additive identity.
   (b) Multiplication is closed, associative and commutative. 1 is the unique multiplicative identity.
   (c) Multiplication distributes over addition.
   (d) Every rational has a unique additive inverse.
   (e) Every non–zero integer has a unique multiplicative inverse.
   (f) There is an order relation ...
   (g) Subtraction is a closed operation.
   (h) Division of a rational by any non-zero rational is possible
   (i) ...... many other derived/inferred properties

4. Despite earlier education on these matters, who can prove:
   (a) -1 times -1 equals $+1$
   (b) $\frac{2}{3} \div \frac{5}{7} = \frac{2 \cdot 7}{3 \cdot 5}$

5. Fundamental drawback: Negative numbers and Rationals were introduced through notation!

6. How to rectify? Study Robert Anderson's Set theory and construction of numbers or equivalents or wikipedia

7. A very brief hint:
   (a) Put equivalence relation on $\mathbb{N} \times \mathbb{N}$ where $(a, b) \sim (c, d)$ if $a + d = b + c$ to get equivalence classes as integers. So, the integer -2 is the equivalence class $[(5, 3)]$
   (b) Put equivalence relation on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ where $(p, q) \sim (r, s)$ if $ps = qr$ to get equivalence classes as rationals. So, the rational $\frac{-2}{3}$ is the equivalence class $[(-2, 3)]$

1. Fundamental Question:

   What is the set of numbers which is sufficient to measure physical quantity like 'Length'? [and likewise Mass and Time]

2. More precisely: What magical set M do we need so that there is a one-to-one correspondence between elements of M and points on a idealized physical straight line L?

   Attempted answers: Naturals, Integers, Rationals ... all necessary but insufficient!

3. Recall that square root of 2 is not a rational number. Inspired by this we have an incomplete answer: Add $\sqrt{2}, \sqrt{3}, \ldots - \sqrt{2}, -\sqrt{3}, \ldots \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, \ldots, \sqrt[3]{2}, \sqrt[3]{3}, \ldots$???

4. Some tools: Pick a stick and call it of 'standard' length, say 1 foot or 1 metre or 1 unit length or merely 1. By trial and error, take two sticks of equal length [equal as far as you can see with eye, magniying glass, microscope etc.] and line them up and match with the standard. Then each of these has length $\frac{1}{2}$. Likewise other fractional lengths. Including lengths of $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

5. Now by experience we can say a measurement of the length ℓ of a stick is between 1 and 2 metres, is between 1.4 and 1.5 metres, is between 1.41 and 1.42 metres, etc. ...

6. Mathematically a *measurement* is an interval $I_1 = [s_1, b_1]$ where s_1 and b_1 are rational numbers and we implicitly assume that $s_1 < b_1$ and we want to indicate that the length ℓ is between the smaller number s_1 and the bigger number b_1.

7. Second measurement is an interval $I_2 = [s_2, b_2]$ where s_2 and b_2 are rational numbers. It is an improvement over the first if and only if $I_1 \supset I_2$.

8. A *lab measurement* for length is thus a finite sequence of intervals with rational endpoints such that $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n$, for some natural number n.

9. What is a perfect measurement for length? Is it an infinte sequence of intervals $[s_1, b_1] \supset [s_2, b_2] \supset [s_3, b_3] \supset \cdots \supset \cdots$, such that:

   (a) $s_1 \leq s_2 \leq s_3 \leq \cdots$ and $b_1 \geq b_2 \geq b_3 \geq \cdots$? –or –

   (b) $s_1 < s_2 < s_3 < \cdots$ and $b_1 > b_2 > b_3 > \cdots$? –or –

   (c) ?

10. A *perfect measurement* for length is an infinte sequence of intervals $[s_1, b_1] \supset [s_2, b_2] \supset [s_3, b_3] \supset \cdots \supset \cdots$, such that:

    (a) For each natural number k, there is a natural number n such that width of $I_n = b_n - s_n < \frac{1}{10^k}$.

    (b) Equivalently, for each positive rational number ϵ, [no matter how small], there there is a natural number n such that width of $I_n = b_n - s_n < \epsilon$.

    (c) Equivalently, "limit" of widths of interval, $\lim_{n\to\infty} \text{width } (I_n) = 0$.

11. Can two perfect measurements represent the same length? If so, under what conditions? Two perfect measurements $I_1 \supset I_2 \supset I_3 \supset \cdots \supset \cdots$ and $J_1 \supset J_2 \supset J_3 \supset \cdots \supset \cdots$ are equivalent if for each natural number n, there is a natural k such that $J_k \subset I_n$ and vice-versa. Equivalently, if $I_n \cap J_n$ is non-empty for each natural n.

12. Real numbers are exactly equivalence classes of perfect measurements. The set of real numbers is denoted by $\mathbb{R}$.

13. Some hints: How to add, subtract, multiply, divide? How to put order relation? How to prove some properties of these operations and of the order relation?

14. Short-cut: Axioms

1. Axioms for the *complete ordered field* $\mathbb{R}$.

   (a) There is a function $f_+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, called the addition function. Assume four axioms about addition, viz., commutativity, associativity, existence of an identity and existence of an additive inverse for each real.
   Inferences:

      i. Prove that there is a unique additive identity. Denote it by 0 and call it *zero*.
      ii. Prove that every real has a unique additive inverse. Denote the additive inverse of a real a by $-a$.

   (b) There is a function $f_\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, called the multiplication function. Assume four axioms about multiplication, viz., commutativity, associativity, existence of an identity and existence of a multiplicative inverse for each real not equal to zero.
   Inferences:

      i. Prove that there is a unique multiplicative identity. Denote it by 1 and call it *one*.
      ii. Prove that every real not equal to zero has a unique multiplicative inverse. Denote the multiplicative inverse of a non–zero real a by $1/a$.

   (c) Multiplication distributes over addition.

   (d) There exists a non–empty subset, $\mathbb{P} \subset \mathbb{R}$, called the set of positive real numbers which is closed under addition and multiplication. Further given any real x, exactly one and no more of the following is true:

      (N) $-x \in \mathbb{P}$
      (Z) $x = 0$
      (P) $x \in \mathbb{P}$.

   Further definitions and inferences:

      i. Given any reals a and b, define $a < b$ if and only if $b + (-a) \in \mathbb{P}$. Say $a \leq b$ if either $a < b$ or $a = b$.
      ii. Given a subset $S \subset \mathbb{R}$, define $u \in \mathbb{R}$ to be an *upper bound* of S if $s \leq u$ for every $s \in S$. We say a set is *bounded above* if it has an upper bound.
      iii. Given a subset $S \subset \mathbb{R}$, define $l \in \mathbb{R}$ to be a *least upper bound* of S if l is an upper bound of S and $l \leq u$ for any upper bound u of S.

   (e) Every non–empty subset of $\mathbb{R}$ which is bounded above has a least upper bound. This is the completeness axiom.

Following is a list of some of the derived properties of $\mathbb{R}$.

2. $a \cdot 0 = 0$ for every $a \in \mathbb{R}$.

3. $a + x = a + y$ implies $x = y$. Corollary: $a + x = a$ implies $x = 0$.

4. $a \cdot x = a \cdot y$, $a \neq 0$ implies $x = y$. Corollary: $a \cdot x = a$, $a \neq 0$ implies $x = 1$.

5. $a \cdot b = 0$ implies $a = 0$ or $b = 0$.

6. Define subtraction of any two reals $a - b := a + (-b)$. It is neither commutative nor associative. However, 0 works as the identity and every element is its own inverse. Addition and subtraction are opposites of each other, viz., $(a + b) - b = a$ and $(a - b) + b = a$ for any reals a and b.

7. Define division of any two reals $a/b := a \cdot \frac{1}{b}$, for $b \neq 0$. It is neither commutative nor associative. 1 works as the identity and every non–zero element is its own inverse. Multiplication and division are opposites of each other, viz., $(a \cdot b)/b = a$ and $(a/b) \cdot b = a$ for any reals a and $b \neq 0$.

8. Multiplication distributes over subtraction and division distributes over both addition and subtraction.

9. The relation $\leq$ is transitive, compatible with addition and compatible with multiplication.

10. For any non–zero $a \in \mathbb{R}$, $0 < a^2$. Corollary: $0 < n$ for every natural n.

11. Archimedean Property viz., for any real number x, there exists a natural N such that $x < N$.

12. Density of Rationals viz., given any two real numbers $x < y$, there exists a rational $x < r < y$.

1. Give a precise definition of the maximum (and the minimum) of a finite collection of reals.

2. Is it legitimate to use the concept of max/min for infinite sets?

   (a) For example, consider the set of rationals $S := \{\frac{n}{n+1} | n \in \mathbb{N}\}$. What is the maximum of S?

   (b) If there is no element $u \in S$ such that $s \leq u$ for all $s \in S$, can you find such a $u \in \mathbb{R}$? Can you find all such u?

   (c) Among all such u that you found, what is special about 1? What is wrong with saying 1 is the minimum of all such u? Can you give a better definition which will pick out such a maximum (more precisely, extended concept of maximum) in all cases?

3. Complete a similar exercise of finding the 'minimum' for the set $T := \{1 + \frac{1}{n} | n \in \mathbb{N}\}$.

4. Definitions of *upper bound*, *lower bound* and *bound* for a subset of reals. Definition of a *bounded* set. Examples.

5. Definition of *supremum* [sup] or *least upper bound* [lub] and *infimum* [inf] or *greatest lower bound* [glb] for a subset of reals. Examples.

6. Existence of sup as guaranteed by the completeness axiom. Can you prove existence of inf using the completeness axiom?

   (a) Write statements of completeness property for naturals, integers and rationals.

   (b) Show that completeness is valid for naturals and integers, but not for rationals.

7. sup and inf of a set, if they exist, are uniquely determined.

8. (a) {Anything less than the supremum is not an upper bound}
   Proposition: If h is the supremum of $S \subset \mathbb{R}$, then for any real $\epsilon > 0$, $h - \epsilon$ is not an upper bound for S.

   (b) {Anything more than the infimum is not a lower bound}
   Proposition: If m is the infimum of $S \subset \mathbb{R}$, then for any real $\epsilon > 0$, $m + \epsilon$ is not a lower bound for S.

9. (a) {If anything less than an upper bound is not an upper bound, then it is the least upper bound}
   Proposition: Let $S \subset \mathbb{R}$ be non-empty and let u be an upper bound for S. If for every real $\epsilon > 0$, there exists a $t \in S$ such that $u - \epsilon < t \leq u$, then $u = \sup S$.

   (b) {If anything more than a lower bound is not a lower bound, then it is the greatest lower bound}
   Proposition: Let $S \subset \mathbb{R}$ be non-empty and let ℓ be a lower bound for S. If for every real $\epsilon > 0$, there exists a $t \in S$ such that $\ell \leq t < \ell + \epsilon$, then $\ell = \inf S$.

10. Proof of Archimedean Property using completeness.

    If there is no natural number bigger than a given real number r_0, the set of naturals is bounded above by r_0. Using completeness, let s be the supremum of naturals. Then, $s - 1$ is not an upper bound for the naturals and hence there is a natural n such that $s - 1 < n$. This implies $s < n + 1$ with $n + 1$ a natural greater than s, the supremeum of naturals: a contradiction.

11. Proof of density of rationals: Read up

12. Discussion topic:

    Archimedean Property can be interpreted simply as: 'there is no greatest real number'.

    Question: Is there a smallest real number?

    More interesting: Is there a smallest positive real number?

    Isn't ∞ the greatest real number?