1. Definition and notation for a sequence.

2. **Clarification:** A list of real numbers $a_1, a_2, \ldots, a_n$ for any $n \in \mathbb{N}$ is called a **finite sequence** of real numbers. Contrast this with $a_1 + a_2 + \cdots + a_n$ is called a **finite sum**.

3. Given a sequence (a_n), there is no concept of the *eventual value* of (a_n) – simply because we have not defined a_∞. However, we have some idea of whether the given sequence (a_n) is or is not getting closer and closer to a fixed real number a. Denote this by $a_n \to a$. Let us attempt to give a criterion which should satisfy the conditions:

   (a) in every example where we believe $a_n \to a$, the criterion should be true,

   (b) in every example where we believe $a_n \not\to a$ the criterion should be false and

   (c) there should be no need to modify the criterion in the face of new examples.

4. (a) Take a small real number, say, $\epsilon_1 = 1$.
   <u>Criterion 1</u>: We say $a_n \to a$ if there exists a natural number N_1 such that $|a_n - a| < \epsilon_1 = 1$ for all $n \geq N_1$. Note that this criterion demands that all but a finite number of terms of the sequence be within a distance of 1 from a.
   This criterion works in proving: (i) every constant sequence $a_n = a$ satisfies $a_n \to a$, (ii) $(\frac{1}{n}) \to 0$ and even (iii) $(-1)^n \not\to -1, 0, 1$. However, consider the sequence $b_n = b + \frac{1}{2} \cdot (-1)^n$ for all n. We do not believe (b_n) is getting closer and closer to b, but Criterion 1 makes $b_n \to b$.

   (b) Perhaps $\epsilon_1 = 1$ is not small enough. Take $\epsilon_2 = \frac{1}{2}$.
   <u>Criterion 2</u>: We say $a_n \to a$ if there exists a natural number N_2 such that $|a_n - a| < \epsilon_2 = \frac{1}{2}$ for all $n \geq N_2$. This criterion works in cases (i)–(iii) listed above. This criterion works in the case of (b_n) given above to show $b_n \not\to b$. So Criterion 2 is better than Criterion 1. However, consider the sequence $c_n = c + \frac{1}{4} \cdot (-1)^n$ for all n. We do not believe (c_n) is getting closer and closer to c, but Criterion 2 makes $c_n \to c$.

   (c) Perhaps $\epsilon_2 = \frac{1}{2}$ is not small enough. Take $\epsilon_3 = \frac{1}{3}$.
   <u>Criterion 3</u>: We say $a_n \to a$ if there exists a natural number N_3 such that $|a_n - a| < \epsilon_3 = \frac{1}{3}$ for all $n \geq N_3$. This criterion works in all the cases Criteria 1 & 2 work given above. It also works to show $c_n \not\to c$. However, consider the sequence $d_n = d + \frac{1}{6} \cdot (-1)^n$ for all n. We do not believe (d_n) is getting closer and closer to d, but Criterion 3 is true here.

   (d) Perhaps $\epsilon_3 = \frac{1}{3}$ is not small enough. Take $\epsilon_0 > 0$ to be some fixed *small* real number.
   <u>Criterion 0</u>: We say $a_n \to a$ if there exists a natural number N_0 such that $|a_n - a| < \epsilon_0$ for all $n \geq N_0$. This criterion works in all cases where Criteria 1–3 work, if $\epsilon_0 < \frac{1}{3}$. Also, one can show $(d_n) \not\to d$ if $\epsilon_0 \leq \frac{1}{6}$. However, consider the sequence $a_n = a + \frac{\epsilon_0}{2} \cdot (-1)^n$ for all n. We do not believe (a_n) is getting closer and closer to a, but Criterion 0 is true here.

5. Observation: Each of the criteria 0–3 has to be necessarily true in the examples we have of sequences *approaching* a real number. Whereas, on the contrary, given any fixed criterion among them, there is an example for a sequence for which the criterion believes that the sequence approaches a real number – while we do not believe this to be so. Moreover, varying the value of ϵ_0, Criterion 0 is actually a collection of infinitely many criteria.

6. Thus we are faced with a situation where infinitely many criteria are necessary for our notion of a sequence getting closer and closer to a real number, whereas, no single one of them is sufficient. Cauchy gathered all the conditions together to capture our notion in the definition below.

7. Cauchy's definition: We say $a_n \to a$ if (and only if) the following is true:
   For any given real $\epsilon > 0$, there exists a natural N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

8. Constant sequence, Tail of a sequence

9. We claim the following limits

   (a) $(\frac{1}{n}) \to 0$

   (b) $(\frac{1}{n^2}) \to 0$

   (c) $(\frac{1}{n^p}) \to 0$ for each fixed natural p

   (d) $(\frac{1}{1+n\alpha}) \to 0$ for each fixed real $\alpha > 0$

   (e) $(b^n) \to 0$ for every fixed real $-1 < b < 1$

   (f) $(c^{1/n}) \to 1$ for every fixed real $c > 0$

   (g) $(n^{1/n}) \to 1$

Brief hints: Given any real $\epsilon > 0$, we have $\frac{1}{\epsilon}$ as a real number and there exists a natural number N such that $\frac{1}{\epsilon} < N$, which implies that for all $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \epsilon$.

$\frac{1}{1+n\alpha} < \frac{1}{n\alpha}$

Since $0 < b < 1$, we can write $b = 1/(1+a)$, where $a := (1/b) - 1$ so that $a > 0$. By Bernoulli's Inequality, we have $(1+a)^n > 1 + na$. Hence $0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$

10. Prove $(-1)^n$ does not converge, *i.e.*, diverges.

11. Prove $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} =?$

12. Template for applying Cauchy's definition to prove $a_n \to a$:

Rough Work		Credit worthy work	
START:	Given a real $\epsilon > 0$	START:	Given a real $\epsilon > 0$
DO SOMETHING:	? ?	SAY:	My N is equal to ...
FIND:	A natural $N =?$	ASSUMING:	$n \geq N$
		DO SOMETHING*	? ?
		GET FINALLY:	$\|a_n - a\| < \epsilon$

1. Uniqueness of limits. If $a_n \to a, b$, then $a = b$.

   Proof: Use the lemma: if a non-negative number is smaller than every positive number, it has to be zero.

   Can you make $|a - b|$ smaller than every positive number?

2. If two sequences converge to the same real number, are the two sequences 'equal'? If two sequences converge to the same real number, do they have to be on 'different' sides of the limit?

3. A sequence $X = (x_n)$ is *bounded* if the set $\{x_n | n \in \mathbb{N}\}$ is bounded or equivalently there exists a real B such that for every natural n, $|x_n| \leq B$. Picture.

4. Why can't one take $B = \max(x_1, x_2, x_3, \ldots)$?

5. Proposition: Convergent implies bounded.

   Proof:

   Method 1 Except for finitely many terms, all others cluster around the limit.

   Method 2 Can you explain when $a_n \not\to a$?

6. Building new sequences from one given sequence (a_n):

   (a) Constant multiple sequence $(c \cdot a_n)$ for some real c
   (b) Square sequence (a_n^2)
   (c) Cube sequence (a_n^3)
   (d) p–th power sequence (a_n^p) for natural p. The latter can be extended to include $p = 0$
   (e) To extend this to all integral p, need to assume none of the $a_n = 0$
   (f) Similarly get fractional powers under additional assumptions, if necessary
   (g) Further, let $f : \mathbb{R} \to \mathbb{R}$ be any polynomial function, viz., $f(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0$ for real numbers $c_0, c_1, \cdots, c_k$ and natural k. Then $(f(a_n))$ is a new sequence
   (h) Fundamental question: If $a_n \to a$, does $f(a_n) \to f(a)$?

7. Building new sequences from two given sequences (a_n) and (b_n):

   (a) their sum $(a_n + b_n)$
   (b) their difference $(a_n - b_n)$
   (c) product $(a_n \cdot b_n)$ and
   (d) quotient (a_n / b_n) [assuming none of the b_n is zero]
   (e) Proposition: If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
   (f) Proposition: If $a_n \to a$ and $b_n \to b$, then $a_n - b_n \to a - b$.
   (g) If $a_n \to a$ and $b_n \to b$, then $a_n \cdot b_n \to a \cdot b$.
   (h) If $a_n \to a$ and $b_n \to b$, and none of the $b_n = 0$ and $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.
   (i) How about proofs of these propositions?

8. Building new sequences from more than two given sequences.

9. Discussion topic:

   (a) Examples for bounded sequences which are not convergent.
   (b) Sequences which seem to have two "limit–like" points? [NOT a formal phrase: don't use it!]
   (c) three limit–like points?
   (d) four limit–like points?,...
   (e) infinitely many?
   (f) all rationals as limit–like points?
   (g) all irrationals?
   (h) all reals???
   (i) any given subset of reals?
   (j) when is a real number c, a "limit-like" point of a given sequence a_n: definition?

1. Proposition: Suppose $a_n \to a$ and for every natural n, $a_n \geq 0$. Then $a \geq 0$.

   Picture and Proof: If $a < 0$, take $\epsilon = -a > 0$ and get a natural N from the definition such that for all $n \geq N$, we have $a - \epsilon < a_n < a + \epsilon$. In particular for $n = N$, we get $a - (-a) < a_N < a + (-a) = 0$ contradicting the hypothesis that $a_N \geq 0$.
   Question: If $a_n \to a$ and for every natural n, $a_n > 0$, then is $a > 0$? If $a_n \geq c$ for every n, then is $a \geq c$? Similar questions with $\leq$, etc..

2. Proposition: Suppose $a_n \to a$ and $b_n \to b$ with $a_n \leq b_n$ for every n. Then, $a \leq b$.

   Picture and Proof: Apply previous proposition to the difference of given sequences.

3. Proposition: Suppose $a_n \to a$ and $\alpha \leq a_n \leq \beta$. Then $\alpha \leq a \leq \beta$.

   Picture and Proof: Apply previous proposition to the constant sequence $b_n = \beta$, etc..

4. Squeeze/Sandwich/Pinching Theorem: For three sequences, $a_n \leq b_n \leq c_n$ with $a_n \to l$ and $c_n \to l$. Then the sequence b_n converges and the limit is l.

   Picture and Proof: Given a real $\epsilon > 0$, find a natural N such that for every $n \geq N$, both $|a_n - a|, |b_n - b| < \epsilon$. Then, $-\epsilon < a_n - a \leq b_n - a \leq c_n - a < \epsilon$ for all $n \geq N$. This proves the required.

5. Nested interval property: Let for each natural n, I_n be an interval of real numbers, *viz.*, $I_n = [a_n, b_n]$ for some real numbers $a_n \leq b_n$. Of course, each such interval is non-empty and bounded. If $I_1 \supset I_2 \supset I_3 \supset \cdots$ and width of $I_n = b_n - a_n \to 0$, then $\cap_1^\infty I_n$ is a set with exactly one real number.

   Is this property true for rational numbers?

6. *Increasing, decreasing* and *monotone* sequences. Additional qualifier: 'strictly'

7. Every increasing sequence is bounded below. There are examples of increasing sequences which are not bounded above. Analogous statements for decreasing sequences.

8. Whereas boundedness for a general sequence does not imply convergence, it does for the restricted class of monotone sequences.

9. Monotone Convergence Theorem: An increasing sequence which is bounded above converges to the supremum of the set formed by the sequence.

   Picture and Proof: Let s be the supremum. Given any real $\epsilon > 0$, recall $s - \epsilon$ is not an upper-bound of the sequence and hence there is a natural N such that $s - \epsilon < a_N$. What can you say about a_n for $n \geq N$? Where are they??

   Analogous statement for decreasing sequences and a proof.

10. Given any strictly increasing sequence of naturals $n_1 < n_2 < n_3 < \cdots$, and a sequence of real numbers (a_n), the sequence (a_{n_k}) is called a subsequence of the given sequence (a_n).

    Examples

11. Proposition: If a sequence $a_n \to a$, then every subsequence $a_{n_k} \to a$.

    Contrapositive gives a divergence criterion.

12. Bolzano–Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

    Picture and Proof: Let $I_1 = [\inf S, \sup S]$, where $S = \{a_n | n \in \mathbb{N}\}$, is the set of terms of the sequence.

    Set $L_2 = [\inf S, \frac{1}{2}(\inf S + \sup S)]$ and $R_2 = [\frac{1}{2}(\inf S + \sup S), \sup S]$. Let $A_2 = \{n | a_n \in L_2\}$ and $B_2 = \{n | a_n \in R_2\}$. At least one of A_2 or B_2 is infinite and if A_2 is infinite, set $I_2 = L_2$ and if not, set $I_2 = R_2$.

    Continue ... and apply nested interval property