1. Infinite series generated by a sequence, terms of the series, partial sums and sequence of partial sums, convergent/divergent series, sum or value of series.

2. Notation: $\sum a_n$ for $\sum_{n=1}^{\infty} a_n$.

   Proof of cooonvergence of geometric series $\sum r^{n-1} = \frac{1}{1-r}$ for $|r| < 1$.

   Proof of divergence for $r \geq 1$: Partial sums are unbounded.

   Proof of divergence for $r = -1$: Odd and even terms of sequence of partial sums have different limits.

   Proof of divergence for $r < -1$: ?

3. Telescoping series $\sum \frac{1}{n(n+1)} = 1$.

   Other examples like $\sum \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

4. n–th term test – necessary but not sufficient for convergence of a series. Proof: $a_n = s_n - s_{n-1}$.

5. As a corollary to monotone convergence theorem: A non–negative series converges if and only if the sequence of partial sums is bounded.
   Applications: 2–series converges and in general p–series for $p > 1$ converges by arguing on s_{2^k-1}.
   Harmonic series diverges by arguing $s_{2^k} \geq 1 + \frac{k}{2}$. By comparison, p–series diverges for $0 < p \leq 1$.

6. Cauchy criterion for series as a direct application of cauchy criterion for convergence.

7. Discussion topic:

   (a) How many examples do you know where a series converges *and* you know the sum?

   (b) Can you find "the answer" to $\sum_1^{\infty} \frac{1}{n^2}$?
   Answer: $\frac{\pi^2}{6}$

   (c) Can you find "the answer" to $\sum_1^{\infty} \frac{1}{n^3}$?
   Answer: "Unknown!": What is "knowing" a real number anyway?
   Only known to be rational

   (d) $\sum_1^{\infty} \frac{1}{n^4}$?
   Answer: Known: find out

   (e) $\sum_1^{\infty} \frac{1}{n^5}$?
   Answer: Not known whether rational or irrational!

   (f) Read about *Basel problem*

1. Given two numbers a and b, they can be added in two ways, viz., $a+b$ and $b+a$. The two answers are equal by commutativity. Given three numbers a, b and c, they can be added in twelve ways, viz., $a+(b+c), a+(c+b), b+(c+a), b+(a+c), c+(a+b), c+(b+a), (a+b)+c, (b+a)+c, (b+c)+a, (c+b)+a, (c+a)+b, (a+c)+b$. These twelve answers are equal by commutativity and associativity.

2. Given any finite collection of real numbers, we expect the different ways of adding them to yield the same answer. Having proved this expectation, the coinciding answer can be defined as the *sum* of these numbers.

3. The axioms for reals provide a 2–input addition function $f_+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The identity function may be taken, artificially, as the 1–input addition function. Inductively, given a natural $k \geq 3$, define a k*–input* addition function from $\mathbb{R}^k$ to $\mathbb{R}$ as a function of the form $f_+(g(*\cdots*), h(*\cdots*))$ for an l–input addition function g and an m–input addition function h such that $l+m=k$ and $l, m \leq k-1$. {Here, the first l components of the k component input are fed as input to g and the remaining m components as input to h.}
   As an exercise, find the number of k–input addition functions. By induction, prove that any two k–input addition functions are equal.

4. Given a k–tuple $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, a *reordering* of x is a k–tuple $x_\varphi = (x_{\varphi(1)}, x_{\varphi(2)}, \ldots, x_{\varphi(k)})$ for a bijection $\varphi : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$.
   As an exercise, find the number of reorderings of a given k–tuple. If f_0 is a k–input addition function, prove that for any two reorderings x and y of each other, the evaluations $f_0(x)$ and $f_0(y)$ are equal.

5. Given k real numbers, a *way* of adding them is an evaluation $f(x)$ of a particular k–input addition function f at x – a particular ordering of the given numbers.
   Find the number of ways of adding k real numbers. Prove that all such ways yield the same answer. This is called the *sum* of the given numbers. The following theorems were necessary in the steps leading up to our definition of sum.

   Finite Regrouping Theorem: Given a k tuple $x \in \mathbb{R}^k$, $f(x) = g(x)$ for any two k–input addition functions f and g.
   Finite Rearrangement Theorem: Given a k–input addition function, f, $f(x) = f(y)$ for any two reorderings x and y of the same k–tuple.

6. Can we extend the above two theorems to infinite series?

7. Devise a definition and find an analogue of the regrouping theorem for infinite series.

8. If $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection, the series $\sum a_{\varphi(n)}$ is called a *rearrangement* of the $\sum a_n$. We explore an analogue of rearrangement theorem for infinite series.

9. We have seen $\sum \frac{1}{n}$ diverges. How about $\sum \frac{1}{2n}$ and $\sum \frac{1}{2n-1}$?

10. Alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges. For a proof, observe that s_{2n} is monotonically increasing and s_{2n+1} is monotonically decreasing. Further, $0 < s_{2n} < s_{2n} + \frac{1}{2n+1} = s_{2n+1} < 1$ which implies that the even and odd partial sums are bounded and hence converge. By squeeze theorem to the same inequality, they converge to the same limit. Finally conclude that the series is convergent.

11. Based on this example, we make the definitions: For a given series $\sum a_n$, if $\sum |a_n|$ converges the series converges *absolutely* and if $\sum a_n$ converges while $\sum |a_n|$ diverges, the series converges conditionally. By Cauchy's criterion, one sees that $\sum |a_n|$ converges implies $\sum a_n$ converges.

12. Rearrangement theorem: If $\sum a_n$ converges absolutely, then every rearrangement of the series converges to the same value. For a proof, let $\sum a_n = a$, s_n and t_n the sequence of partial sums of the given series and its rearrangement. Pick a natural N such that for $n \geq N$, both $|s_n - a| < \frac{1}{2}\epsilon$ and $\sum_N^\infty |a_n| < \frac{1}{2}\epsilon$ are true. Let M be a natural number such that the terms $a_1, a_2, \ldots a_N$ appear in the rearranged partial sums t_n for $n \geq M$. Naturally, $M \geq N$. Then, for $n \geq M \geq N$, we have $|t_n - a| \leq |t_n - s_n| + |s_n - a| \leq \sum_N^\infty |a_n| + \frac{1}{2}\epsilon < \epsilon$. Done.

13. Demonstrate that the alternating harmonic series can be rearranged to converge to any real number.

14. Statement of Riemann's rearrangement theorem.