1. Alternating Series Test: If (z_n) is monotonically decreasing sequence of non–negatives converging to zero, the series $\sum(-1)^{n+1}z_n$ converges. Proof is analogous to the example of alternating harmonic series. One just observes that the even subsequence of partial sums s_{2n} is monotonically increasing and the odd subsequence of partial sums s_{2n+1} is monotonically decreasing. Further $0 \le s_{2n} \le s_{2n} + z_{2n+1} = s_{2n+1} \le z_1$. This makes the two subsequences bounded and hence convergent. Using $z_{2n+1} \to 0$ and squeeze theorem they converge to the same value. Using this establish that the alternating series converges.

2. Comparison test for non–negative series. Similar one for negative series. Statement:
   Let $0 \le x_n \le y_n$ for $n \ge K$, some natural. Then (i) $\sum y_n$ converges implies $\sum x_n$ converges. Also, (ii) $\sum x_n$ diverges implies $\sum y_n$ diverges.
   How about if $\sum x_n$ converges or $\sum y_n$ diverges? Further, in above comparison test, do we need $0 \le x_n$? Will the test be true if this condition is dropped?

3. Limit comparison test: (x_n) and (y_n) are positive sequences and $r = \lim \frac{x_n}{y_n}$ exists. Then:
   (i) If $r \neq 0$, $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.
   Proof: Take $\epsilon = \frac{1}{2}r$. There exists a natural K such that $n \ge K$ implies $|\frac{x_n}{y_n} - r| < \frac{1}{2}r$, i.e., $\frac{1}{2}r < \frac{x_n}{y_n} < \frac{3}{2}r$ whence $(\frac{1}{2}r)y_n < x_n < (\frac{3}{2}r)y_n$. Now apply comparison test.
   (ii) If $r = 0$, and $\sum y_n$ is convergent then $\sum x_n$ is convergent.
   Proof: Take $\epsilon = 1$. There exists a natural K such that $n \ge K$ implies $-1 < 0 < \frac{x_n}{y_n} < 1$ whence $0 < x_n < y_n$. Now apply comparison test.

4. Examples: $\sum \frac{1}{n^2+n+1}$ by comparison with $\sum \frac{1}{n^2}$. Limit comparison of $\sum \frac{1}{n^2-n+1}$ with $\sum \frac{1}{n^2}$. Limit comparison of $\sum \frac{1}{\sqrt[3]{n+9}}$ with $\sum \frac{1}{\sqrt[3]{n}}$

5. Value Root Test: Let (a_n) be a sequence of reals.
   Suppose for some real r, $|a_n|^{\frac{1}{n}} \le r$ for all $n \ge K$ for some natural K. If $r < 1$, the series $\sum a_n$ (and $\sum |a_n|$) are convergent. For a proof, compare with geometric series.
   Suppose $|a_n|^{\frac{1}{n}} \ge 1$ for all $n \ge K$ for some natural K. Then the series $\sum a_n$ (and $\sum |a_n|$) are divergent. For a proof, use n–th term test.

6. Limit Root Test: Suppose (a_n) be a sequence of reals such that $r = \lim |a_n|^{\frac{1}{n}}$ exists.
   If $r < 1$, then $\sum a_n$ (and $\sum |a_n|$) are convergent.
   If $r > 1$, then $\sum a_n$ (and $\sum |a_n|$) are divergent.

7. Value Ratio Test: Let (a_n) be a sequence of non–zero reals.
   Suppose that for some real r, $\left|\frac{a_{n+1}}{a_n}\right| \le r$ for all $n \ge K$ for some natural K. If $r < 1$, the series $\sum a_n$ (and $\sum |a_n|$) are convergent. For a proof, compare with geometric series.
   Suppose $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge K$ for some natural K. Then, the series $\sum a_n$ (and $\sum |a_n|$) are divergent. For a proof, use n–th term test.

8. Limit Ratio Test: Suppose that (a_n) is a sequence of non–zero reals such that $r = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists.
   If $r < 1$, then $\sum a_n$ (and $\sum |a_n|$) are convergent.
   If $r > 1$, then $\sum a_n$ (and $\sum |a_n|$) are divergent.