Mean Value Theorem for integrals: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function. Then there exists $\xi \in [a, b]$ such that

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)dx.$$

First Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(x)dx.$$

Then, F is uniformly continuous on $[a, b]$, differentiable on (a, b), and

$$F'(x) = f(x) \qquad \text{for all } x \in (a, b).$$

Let $f : [a, b] \to \mathbb{R}$ be a function. A function F is called an antiderivative or primitive of f if $F'(x) = f(x)$ for all $x \in [a, b]$.

Second Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function and let G be an antiderivative of f. Then,

$$\int_a^b f(x)\, dx = G(b) - G(a).$$

Remark: The theorem holds even if f is not assumed to be continuous.
Hint: If $F = \int_a^x f$ then $F' - G' = 0$ and hence $F - G$ is a constant function.

Taylor's Theorem

Let $f : [a, b] \to \mathbb{R}$ be such that $f, f', f'', \dots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Let $x_0 \in [a, b]$.
Then for any $x \in [a, b]$ there exists $c \in (x, x_0)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

In particular, there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Remark: If $x_0 < x$, then the interval should be taken as (x_0, x).

Power Series

Let (a_n) be a sequence. Then for $x \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} a_n x^n$ is called a power series.

In general the series for $a \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} a_n (x-a)^n$ is called power series around a.

We will assume that $a = 0$.

Theorem: Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges at some $x = x_0$ and diverges at $x = x_1$. Then

1. $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $|x| < |x_0|$.
2. $\sum_{n=0}^{\infty} a_n x^n$ diverges for all $|x| > |x_1|$.

Thus either the series $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$ or there exists unique $r > 0$, such that the series converges absolutely for all $|x| < r$ and diverges for all $|x| > r$.
This r is called the radius of convergence.

If the series converges only at $x = 0$, then the radius of convergence is 0.
If the series converges for all $x \in \mathbb{R}$, then the radius of convergence is ∞.
Formula for radius of convergence:

$$r = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

Although, lim sup has not been discussed in the class, in the special case, when $\lim \sqrt[n]{|a_n|}$ exists, it is known that $\limsup \sqrt[n]{|a_n|} = \lim \sqrt[n]{|a_n|}$. You may use this special case for the calculation of radii of convergence of power series.

Conventions:

- If $\sqrt[n]{|a_n|}$ is a monotonic and unbounded sequence, then we say $r = 0$
- If $\lim \sqrt[n]{|a_n|} = 0$, then we say $r = \infty$

Taylor's series

The power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called Taylor's series of f around a.
If $a = 0$, then the power series is called Maclaurin series.
Remark: If f is infinite times differentiable at a then the corresponding Taylor series is defined. Moreover, $P_n(x)$ is the n-th partial sum of the Taylor series.

Examples

Let $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1-x}$. Then the Taylor's series of f around 0 (i.e. Maclaurin's series) is the geometric series

$$\sum_{n=0}^{\infty} x^n.$$

This converges for all $x \in (-1, 1)$ and diverges for $|x| > 1$. Thus the radius of convergence is 1.

Examples

- $\sum_{n=1}^{\infty}(nx)^n$. In this case, $\sqrt[n]{|a_n|} = n$, which is monotonic and unbounded. Therefore, radius of convergence $r = 0$.
- $\sum_{n=1}^{\infty}(3x)^n$. In this case, $\sqrt[n]{|a_n|} = 3$, which is a constant sequence, hence convergent. Therefore radius of convergence $r = \frac{1}{3}$.
- $\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n$. In this case, $\sqrt[n]{a_n} = \frac{1}{n}$, which converges to 0. Therefore the radius of convergence is $r = \infty$.

Examples

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x > 0.$$

What is the radius of convergence?