Note on Grasping Opposites

opposite In this course we shall encounter adjectives like: bounded below, bounded above, bounded, convergent, cauchy, absolutely convergent, conditionally convergent, increasing, decreasing, monotonic, strictly increasing, strictly decreasing, strictly monotonic, continuous, uniformly continuous, differentiable, integrable. The question addressed in this note is: What are the opposites of these adjectives?

bounded A set $S \subset \mathbb{R}$ is *bounded below* if there exists a $l \in \mathbb{R}$ such that $l < x$ for every $x \in S$.

What is the opposite of "bounded below"?

Poor man's Answer: A set $S \subset \mathbb{R}$ is *not bounded below* if there is no lower bound for S.

Lame Answer: A set $S \subset \mathbb{R}$ is *not bounded below* if there does NOT exist a $l \in \mathbb{R}$ such that $l < x$ for every $x \in S$.

The poor man's answer is correct but cannot be immediately applied to all real numbers and check them to be not lower bounds for a given set. The Lame Answer just states the opposite of the definition. It is correct, but perhaps not immediately useful in proving a particular subset is not bounded below.

A set $S \subset \mathbb{R}$ is *un–bounded below* if for every natural number n, there exists a $s_n \in S$, such that $s_n < -n$.

A set $S \subset \mathbb{R}$ is *unn–bounded below* if there is a sequence (s_n) such that $s_n \in S$ and $s_n - s_{n+1} > 1$.

Can you prove that the definitions for not bounded below, un–bounded below and unn–bounded below are all equivalent?

convergent A sequence $a_n \not\to a$ if there exist a real number $\epsilon_0 > 0$ and a subsequence a_{n_k} such that $|a_{n_k} - a| \geq \epsilon_0$. Why?

increasing A sequence (a_n) is not increasing if there exist natural numbers $l < m$ such that $a_l > a_m$. Why?

monotonic A sequence (a_n) is not monotonic if there exist natural numbers $i < j, l < m$ such that $a_i < a_j$ and $a_l > a_m$. Why?

strictly monotonic A sequence (a_n) is not strictly monotonic if .... complete the sentence.

cauchy A sequence (a_n) is not cauchy if there exists a real number $\epsilon_0 > 0$ and two subsequences (a_{n_k}) and (a_{m_k}) satisfying $|a_{m_k} - a_{n_k}| \geq \epsilon_0$, for all k. Why?

monotonic The purpose of this section is to figure out the "opposite" of a strictly monotonic function.

A function $f : \mathbb{R} \to \mathbb{R}$ is *strictly monotonic* if EITHER for every $x, y \in \mathbb{R}$ with $x < y$, we have $f(x) < f(y)$ OR for every $x, y \in \mathbb{R}$ with $x < y$, we have $f(x) > f(y)$.

Main Question: When is a function $f : \mathbb{R} \to \mathbb{R}$ *not* strictly monotonic?

Answer: A function $f : \mathbb{R} \to \mathbb{R}$ *not* strictly monotonic if one of the following conditions is TRUE

1. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha < \beta < \gamma$ and $f(\alpha) \leq f(\beta) \geq f(\gamma)$

2. There exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha < \beta < \gamma$ and $f(\alpha) \geq f(\beta) \leq f(\gamma)$

Proof: For a function to be NOT strictly monotonic both the conditions below fail:

1. for every $x, y \in \mathbb{R}$ with $x < y$, we have $f(x) < f(y)$
2. for every $x, y \in \mathbb{R}$ with $x < y$, we have $f(x) > f(y)$

Therefore we conclude both the conditions below hold true:

1. there exist $d_1, d_2 \in \mathbb{R}$ with $d_1 < d_2$, we have $f(d_1) \geq f(d_2)$
2. there exist $i_1, i_2 \in \mathbb{R}$ with $i_1 < i_2$, we have $f(i_1) \leq f(i_2)$

Now the order relationship between the four points d_1, d_2, i_1, i_2 falls into the following cases:

1. $d_1 < d_2 \leq i_1 < i_2$
   (a) If $f(d_2) \geq f(i_1)$: Take $\alpha = d_2, \beta = i_1, \gamma = i_2$ and we have $f(\alpha) \geq f(\beta) \leq f(\gamma)$
   (b) If $f(d_2) < f(i_1)$; Take $\alpha = d_1, \beta = d_2, \gamma = i_1$ and we have $f(\alpha) \geq f(\beta) \leq f(\gamma)$
2. $d_1 \leq i_1 < d_2 \leq i_2$
   (a) If $f(i_1) \geq f(d_1)$: Take $\alpha = d_1, \beta = i_1, \gamma = d_2$ and we have $f(\alpha) \leq f(\beta) \geq f(\gamma)$
   (b) If $f(d_1) \geq f(i_1) \geq f(d_2)$; Take $\alpha = i_1, \beta = d_2, \gamma = i_2$ and we have $f(\alpha) \geq f(\beta) \leq f(\gamma)$
   (c) If $f(d_2) \geq f(i_1)$: Take $\alpha = d_1, \beta = i_1, \gamma = d_2$ and we have $f(\alpha) \geq f(\beta) \leq f(\gamma)$
3. $d_1 \leq i_1 < i_2 < d_2$
4. $i_1 \leq d_1 < d_2 \leq i_2$
5. $i_1 \leq d_1 \leq i_2 < d_2$
6. $i_1 < i_2 \leq d_1 < d_2$

Complete the remaining cases yourself.