Note on max, min, lower-bound, upper-bound, least-element, greatest-element, inf & sup

max The purpose of this section is to develop a definition of maximum of a finite subset of real numbers. First we shall define $\max_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Given any two elements x and y in $\mathbb{R}$, define

$$\max_2(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x < y \end{cases} \tag{1}$$

Notice that '$\max_2$' is a 2 input, 1 output function. So far, we have defined 'max' or 'maximum' of two numbers.

Can we extend the definition to a finite set? Suppose $x_1, x_2, x_3, \ldots, x_n$ is a finite-sequence of real numbers, for some natural number n. What is a definition for $\max(x_1, x_2, x_3, \ldots, x_n)$? Saying 'it is the biggest one' or 'largest number in the set' etc. – they are essentially meaningless. They DO NOT constitute a mathematical definition.

Proper definition for maximum of a tuple (ordered sequence) of real numbers: We define $\max(x_1) = x_1$ and $\max(x_1, x_2) = \max_2(x_1, x_2)$ as above. And having defined $\max(x_1, x_2, x_3, \ldots, x_k)$ for a natural number k, we define

$$\max(x_1, x_2, x_3, \ldots, x_{k+1}) = \max_2(\max(x_1, x_2, x_3, \ldots, x_k), x_{k+1}) \tag{2}$$

By the principle of mathematical induction we have defined maximum of any finite ordered collection of real numbers.

Exercise: Prove that the above definition does not depend on the order in which the elements of a finite set are written. With this, maximum of a finite subset of real numbers becomes well–defined.

Elaboration: To help you understand the definition, let us try what we mean by $\max(x_1, x_2, x_3)$, for instance. Following above..

$$\begin{aligned}
\max(x_1, x_2, x_3) &= \max_2(\max(x_1, x_2), x_3) && (3) \\
&= \max_2(\max_2(x_1, x_2), x_3) && (4) \\
&= \begin{cases} \max_2(x_1, x_3) & \text{if } x_1 \geq x_2 \\ \max_2(x_2, x_3) & \text{if } x_1 < x_2 \end{cases} && (5) \\
&= \begin{cases} x_1 & \text{if } x_1 \geq x_2 \text{ and if } x_1 \geq x_3 & \text{(i)} \\ x_3 & \text{if } x_1 \geq x_2 \text{ and if } x_1 < x_3 & \text{(ii)} \\ x_2 & \text{if } x_1 < x_2 \text{ and if } x_2 \geq x_3 & \text{(iii)} \\ x_3 & \text{if } x_1 < x_2 \text{ and if } x_2 < x_3 & \text{(iv)} \end{cases} && (6)
\end{aligned}$$

Alternately, note that the order-relations between three real numbers falls under one of the following 6 cases:

1. $x_1 \geq x_2 \geq x_3$ This falls under (i) above
2. $x_1 \geq x_3 \geq x_2$ This falls under (i) above
3. $x_2 \geq x_1 \geq x_3$ This falls under (iii) above
4. $x_2 \geq x_3 \geq x_1$ This falls under (iii) above
5. $x_3 \geq x_1 \geq x_2$ This falls under (ii) above
6. $x_3 \geq x_2 \geq x_1$ This falls under (iv) above

You could try calculating $\max(3, 4, 5) = \max_2(\max_2(3, 4), 5) = \max_2(4, 5) = 5$, as expected. Likewise, try: $\max(4, 3, 5)$, $\max(5, 3, 4)$ etc. for practice, applying the elaboration above. For further practice, write out $\max(x_1, x_2, x_3, x_4)$, with all sub-cases in full like above.

min Carefully develop a definition for minimum of a finite set completely analogous to the above discussion and definition of max of a finite set.

examples

1. Given the set $F = \{\frac{n}{n+1} | n \in \mathbb{N}\}$, what is $\max F$? The correct answer is: "So far, max has been defined only for finite sets. Unless the definition of max is extended to be applicable to infinite sets, we cannot answer this question." What is $\min F$? The correct answer is "So far, min has been defined only for finite sets. Unless, etc...."
2. Given the set $G = \{z | z \in \mathbb{Z}, z < 0\}$ what is the $\max G$, $\min G$? The correct answer is "max and min of G cannot be determined as G is an infinite set and the definitions are applicable for finite sets alone."

Is 1 NOT the minimum of $\mathbb{N}$? No. Is -1 NOT the maximum of G? Again, no. But for a clarification, see definition of "greatest-element" below. This issue has contributed to some confusion.

lower-bound A real number l is a lower bound for a set $S \subset \mathbb{R}$, if $l \leq s$ for all $s \in S$. Note that it is not necessary for l to be in S.

upper-bound A real number u is an upper bound for a set $S \subset \mathbb{R}$, if $s \leq u$ for all $s \in S$.

examples For the set F above, check that $-10, -1, 0, \frac{1}{4}$ are all lower-bounds while $10, 2, 1.5, 1$ are all upper-bounds. For the set G above, check that $-1, 0, 1, 10, \sqrt{2}$ are all upper-bounds. Further check that G has no lower-bound.

least-element A real number t is a(the?) least-element for a set $S \subset \mathbb{R}$, if (i) $t \in S$ and (ii) $t \leq s$ for every $s \in S$. Note that(can you check and confirm?) t is a least-element for a set S if t is a lower-bound and t is an element of S.

greatest-element A real number g is a(the?) greatest-element for a set $S \subset \mathbb{R}$, if (i) $g \in S$ and (ii) $s \leq g$ for every $s \in S$. Note that(can you check and confirm?) g is a greatest-element for a set S if g is an upper-bound and g is an element of S.

existence It is not necessary for every subset of reals to have a least-element. However, every finite subset of reals has a least-element, the mimimum of the set. Likewise, it is not necesary for every subset of reals to have a greatest-element. Further, every finite subset of reals has a greatest-element, the maximum of the set.

uniqueness If a subset of reals has a least-element, it is unique. Likewiese, if a subset of reals has a greatest-element, it is unique.

examples Check that $\frac{1}{2}$ is the least-element for S, above. Further check that G has no least-element. Show that the set F above has no greatest-element while the set G has -1 as its greatest-element. Owing to the latter, many students want to say the maximum of G is -1 – which is technically incorrect – as maximum has not been defined for an infinite subset of real numbers.

inf, sup For a subset S of real-numbers, let s be an upper bound. If for any upper-bound u of S, we have the property that $s \leq u$, then s is a supremum of S, denoted by $\sup S$. Analogously define infimum of a subset of reals.

existence If S is a non–empty subset of $\mathbb{R}$, and if S has a lower-bound, $\inf S$ exists in $\mathbb{R}$. Likewise if S is a non–empty subset of $\mathbb{R}$, and if S has an upper-bound, $\sup S$ exists in $\mathbb{R}$. The former is the completeness axiom and the latter can be derived from the former.

uniqueness If the supremum of a set exists, it is unique. Likewise for the infimum. Prove these statements.

examples Further, check that $\frac{1}{2}$ is the infimum of F, 1 is the supremum of F, the infimum of G does not exist and -1 is the supremum of G.

Exercises

1. If S is a finite subset of reals, $\min S$ and $\max S$ exist and satisfy $\min S = \text{least} - \text{element of } S = \inf S$, $\max S = \text{greatest} - \text{element of } S = \sup S$.

2. Find examples for a subset S of $\mathbb{R}$ which has no upper-bound. Likewise find examples with no lower-bound.
3. If a set has a lower-bound in reals, it has infinitely many lower-bounds. Likewise for upper bounds.
4. When a least element for a set exists, it is unique. Likewise for greatest-element.
5. Find examples for S a subset of $\mathbb{R}$, which has and does not have the least-element and which has and does not have the greatest-element.
6. Find examples for a subset S of reals having a lower-bound without having a least-element. Likewise, find examples for a subset S having an upper-bound without having a greatest-element.
7. If α is the least element of a subset S of $\mathbb{R}$, then $\alpha = \inf S$. Likewise, If β is the greatest-element of a subset S of $\mathbb{R}$, then $\beta = \sup S$.
8. Find examples for a subset S of reals having no $\inf S$ in $\mathbb{R}$. Likewise, find examples for a subset of reals with no $\sup S$ in reals.
9. Find examples for a subset S of $\mathbb{R}$ such that $\inf S$ exists but S does not have a least-element. Likewise, find examples for sets which have a well–defined supremum but not greatest-element.
10. From your exercises conclude that the concepts of min, least-element, infimum are of 'increasing' generality. Likewise, conclude that max, greatest-element, supremum are of 'increasing' generality. Finally, you might take this observation as "justifying" the need for our lectures on supremum and infimum as opposed to "being-stuck" with maximum and minimum.