1. Prove that the Principle of Mathematical Induction implies and is implied by The Well–Ordering Principle.

Hint:

Assume Induction is a valid principle. Let S be a non–empty subset of natural numbers which does not have a least element. Suppose $1 \in S$. Then 1 is the least element of S contradicting hypothesis. Conclude $1 \notin S$.

Assume natural numbers $1, 2, \ldots, k \notin S$. Then, if $k+1 \in S$, $k+1$ would be the least element of S, contradicting hypothesis. Conclude $k+1 \notin S$.

From Induction, no natural number n is in S. Thus, S is empty, contradicting hypothesis. Consequently Well–ordering is valid.

Next, assume Well–ordering is a valid principle. Suppose a proposition $P(1)$ is true and assuming $P(k)$ is true for some natural k, we have established that $P(k+1)$ is true. We have to establish that $P(n)$ is true for all natural n. Let S be the subset of all natural n for which $P(n)$ is false. Assume S is non–empty.

From Well–ordering, there is a minimal element $k_{\min} \in S$. $k_{\min} \neq 1$ (as $P(1)$ is true) and hence $k_{\min} - 1$ is a natural number. Further $k_{\min} - 1 \notin S$, by property of $k_{\min}$. Thus, $P(k_{\min} - 1)$ is true and by the hypothesis on induction, $P(k_{\min})$ is true, contradicting $k_{\min} \notin S$. Thus $S \neq \phi$ is an incorrect assumption.

2. (a) For any two real numbers z and a, if $z + a = a$, then $z = 0$.
   (b) For any real number a, prove: $a \cdot 0 = 0$.
   (c) (Additive Cancellation Law) For any three real numbers a, x, y, if $a + x = a + y$, then show that $x = y$.
   (d) (Multiplicative Cancellation Law) For any three real numbers a, x, y, if $a \cdot x = a \cdot y$, and $a \neq 0$, then show that $x = y$.

Hint:

(a)

$$\begin{aligned} z = {} & z + 0 && \text{By Axiom 1(a) for } \mathbb{R}, 0 \text{ is additive identity} \\ = {} & z + (a + (-a)) && \text{Additive inverse, Axiom 1(a)} \\ = {} & (z + a) + (-a) && \text{Associativity of addition, Axiom 1(a)} \\ = {} & (a) + (-a) && \text{Hypothesis} \\ = {} & 0 && \text{Axiom 1(a)} \end{aligned}$$

If you find quoting Axiom 1(a), Axiom 2(b), etc. difficult, you should write a minimum of:

$$\begin{aligned} z = {} & z + 0 && 0 \text{ is additive identity} \\ = {} & z + (a + (-a)) && a \text{ and its additive inverse add to zero} \\ = {} & (z + a) + (-a) && \text{Associativity of addition} \\ = {} & (a) + (-a) && \text{Hypothesis} \\ = {} & 0 && a \text{ and its additive inverse add to zero} \end{aligned}$$

This way of writing a proof ensures that you are referring to suitable axioms and previously proven results. You are encouraged to practice this for all tutorial questions.

Question: What is wrong with saying: Cancel a on both sides to get $z = 0$?

(b)

$$\begin{aligned} a \cdot 0 + a = {} & a \cdot 0 + a \cdot 1 && \text{? Why} \\ = {} & a \cdot (0 + 1) && \text{? Why} \\ = {} & a \cdot 1 && \text{? Why} \\ = {} & a && \text{? Why} \end{aligned}$$

From above, $a \cdot 0 + a = a$. Now we conclude $a \cdot 0 = 0$. (? Why) The student should provide answers to ‘Why-s’ rather than leaving them for evaluators on a test.

(c) Add $(-a)$ to both sides of the equation, use the definition of additive-inverse to get the required result.

Question: Do you need the ancient Euclid's common notion 'When equals are added to equals, the results are equal'? Further is this 'common notion' true? Is a proof necessary, for this notion or unnecessary as it is 'common'? Do you have a proof?

(d) Since $a \neq 0$, multiply both sides of the equation by the multiplicative inverse $1/a$, which exists by the axiom 1(b).

Question: Similar to previous part.

3. If $a, b \in \mathbb{R}$, prove the following.

(a) $a+b = 0 \Rightarrow b = -a$ (b) $-(-a) = a$ (c) $(-1) \cdot a = -a$ (d) $(-1) \cdot (-1) = 1$.

Hint:

(a) By the commutativity axiom of addition, $a+b = b+a$. Using this with the hypothesis, $a+b = 0 = b+a$. The latter equations are the defining equations for the additive inverse of a. Hence $b = -a$.
Alternately, given $a + b = 0$, add $(-a)$ to both sides to get $(a + b) + (-a) = 0 + (-a) \ldots (1)$. By associativity of addition, commutativity of addition, associativity again, definition of additive inverse and zero as additive identity, we have the equalities $a+(b+(-a)) = a+((-a)+b) = (a+(-a))+b = 0 + b = b$ as the left hand side of (1). However, by the definition of zero, the right hand side of (1) is $-a$.
This method uses Euclid's common notion: 'Equals added to equals yields equals – i.e., if $x = y$ for two reals x and y, then $x + c = y + c$ for any real c. This property is not explicitly listed as an axiom for reals. Is it necessary to prove this?
What is deficient with the method: $a + b = 0 \Rightarrow a + b - a = 0 - a \Rightarrow b = -a$? Or with the remark that $a + b = 0 \Rightarrow b = -a$ by the definition of additive inverse?

(b) By definition of additive inverse, $(-a) + a = 0$. Using (a) on this, $a = -(-a)$.
More elaborately, definition of additive inverse $a + b = b + a = 0$ is symmetric in a and b. That is the definition allows us to say $a = -b$ and $b = -a$ from the two equations. Consequently, the second conclusion $b = (-a)$ in the first yields $a = -(-a)$.
What is terrible with $-(-a) = (- \cdot -)a = +a = a$ using minus $\times$ minus $= +$?

(c) By definition of additive inverse we have $1 + (-1) = 0$. Multiplying by a, $1 \cdot a + (-1) \cdot a = 0 \cdot a$. By definition of multiplicative identity, $1 \cdot a = a$ and Theorem 2.1.2(c) in our text BaSh, $0 \cdot a = 0$. Consequently, $a + (-1) \cdot a = 0$. And, likewise, $(-1) \cdot a + a = 0$. The latter two equations imply $(-1)a = -a$, by definition of additive inverse.

(d) Parts (b) and (c) imply (d) for $a = (-1)$.

4. Prove that if $a, b \in \mathbb{R}$, then

(a) $-(a + b) = (-a) + (-b)$
(b) $(-a) \cdot (-b) = a \cdot b$
(c) $1/(-a) = -(1/a)$, if $a \neq 0$
(d) $-(a/b) = (-a)/b$, if $b \neq 0$

Hint:

(a) $((-a)+(-b))+(a+b) = (a+b)+((-a)+(-b)) = ((a+b)+(-a))+(-b) = (a+(b+(-a)))+(-b) = (a + ((-a) + b)) + (-b) = ((a + (-a)) + b) + (-b) = (0 + b) + (-b) = b + (-b) = 0$.

(b) Use $-a = (-1)a$, $-b = (-1)b$, properties of multiplication and previous problem 1.

(c) By using 1(c), $(-(\frac{1}{a})) \cdot (-a) = ((-1) \cdot \frac{1}{a}) \cdot ((-1) \cdot a)$. Using associativity and 1(d), the latter reduces to $\frac{1}{a} \cdot a = 1$.

(d) Similar to 2(c).

5. Read about properties of $\mathbb{R}$ from the text BaSh. Let a, b, c and d be reals.
   (a) If $a < b$ and $c < d$, prove that $a + c < b + d$.
   (b) If $0 < a < b$ and $0 < c < d$, prove that $0 < ac < bd$.

Hint:

(a) By Theorem 2.1.7(b) of BaSh, $a + c < b + c < b + d$.

(b) By the same theorem, $0 \cdot c = 0 < ac < bc < bd$.

6. Give examples for

   (a) $S \subseteq \mathbb{Q}$ such that $\sup S \in \mathbb{Q}$
   (b) $T \subsetneq \mathbb{Q}$ such that $\sup T \notin \mathbb{Q}$
   (c) $U \subseteq \mathbb{R} \setminus \mathbb{Q}$ such that $\sup U \in \mathbb{R} \setminus \mathbb{Q}$
   (d) $V \subsetneq \mathbb{R} \setminus \mathbb{Q}$ such that $\sup V \notin \mathbb{R} \setminus \mathbb{Q}$.

Hint:

(a) Take $S = \{-\frac{1}{n} | n \in \mathbb{N}\} \subset \mathbb{Q}$ with $\sup S = 0 \in \mathbb{Q}$.

(b) Take $T = \{x_n | n \in \mathbb{N}\} \subset \mathbb{Q}$, where (x_n) is defined recursively as follows. $x_1 = 1$ and having defined x_k for any natural k, define $x_{k+1} = x_k + \frac{p}{10^k}$ for that unique $p \in \{0, 1, 2, \ldots, 9\}$ satisfying $x_{k+1}^2 < 2 < (x_{k+1} + \frac{1}{10^k})^2$ with $\sup T = \sqrt{2} \notin \mathbb{Q}$.

(c) Take $U = \{\sqrt{2} - \frac{1}{n} | n \in \mathbb{N}\} \subset \mathbb{R} \setminus \mathbb{Q}$ with $\sup U = \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

(d) Take $V = \{-\frac{\sqrt{2}}{n} | n \in \mathbb{N}\} \subset \mathbb{R} \setminus \mathbb{Q}$ with $\sup V = 0 \notin \mathbb{R} \setminus \mathbb{Q}$.

7. If $S = \{\frac{1}{n} - \frac{1}{m} | n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$. Justify.

Hint:
Get Started with a few terms like

$$0 = \frac{1}{1} - \frac{1}{1} = \frac{1}{2} - \frac{1}{2} = \frac{1}{3} - \frac{1}{3} = \cdots$$
$$\frac{1}{2} = \frac{1}{1} - \frac{1}{2}, \quad \frac{2}{3} = \frac{1}{1} - \frac{1}{3}, \quad \frac{3}{4} = \frac{1}{1} - \frac{1}{4}, \ldots$$
$$-\frac{1}{2} = \frac{1}{2} - \frac{1}{1}, \quad \frac{1}{6} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{4} = \frac{1}{2} - \frac{1}{4} \ldots$$

Verify that for any natural numbers m and n, the following holds

$$-1 < \frac{1}{n} - \frac{1}{1} \leq \frac{1}{n} - \frac{1}{m} \leq \frac{1}{1} - \frac{1}{m} < +1.$$

Thus†, -1 is a lower bound and 1 is an upper bound. Given any real $\epsilon > 0$, there exists‡ natural numbers N and M such that $\frac{1}{N} - \frac{1}{1} < -1 + \epsilon$ and $\frac{1}{1} - \frac{1}{M} > 1 - \epsilon$ which shows that any number greater than -1 is NOT a lower bound while any number smaller than $+1$ is NOT an upper bound. Conclude: $-1 = \inf S$, the greatest of the lower bounds and $+1 = \sup S$, the least of the upper bounds.

†: Why? How can you conclude about bounds for S from the given inequalities?

‡: Why? How do you know such N and M exist?

$\urcorner$ Flagging $\dagger$ and $\ddagger$ are examples of critical thinking in Mathematical Real Analysis being showcased in MA-101.

8. Let $S \subset \mathbb{R}$. Suppose $\phi \neq T \subset S$, such that T has an upper bound in S. If for every such T, there is a † least upper-bound in S, we say S has LUB property.
   (a) Show that $\mathbb{N}$ & $\mathbb{Z}$ have LUB property.
   (b) Show that $\mathbb{Q}$ doesn't have LUB property.

(c) Give an example for S such that $\mathbb{Z} \subsetneq S \subsetneq \mathbb{R}$ which has LUB property. Justify.

(d) Show that $\mathbb{R} \setminus \{0\}$ does NOT have the LUB property.

†: The indefinite article "a" in English denotes "one". In mathematics, it means the existence of at least one such element.

Hint:

(a) For any non–empty $T \subset \mathbb{N}$ which is bounded above, let $U_T = \{b | b \in \mathbb{N},\ t \leq b,\ \text{for all } t \in T\}$, the collection of all upper–bounds of T. The set U_T is non-empty by hypothesis and has a least element, say, b_T by well–ordering principle of $\mathbb{N}$. This b_T is the least element of U_T and hence the lub of T.

The Well–Ordering Principle is false for $\mathbb{Z}$. Can you prove the following?

> Well–Ordering Principle* for $\mathbb{Z}$. If B is any non-empty subset of $\mathbb{Z}$ bounded below, then B has a least element.
>
> Hint-of-a-proof: If $B \subset \mathbb{N}$, use the Well–Ordering Principle on $\mathbb{N}$. If not, let b be a lower bound on B. Define the function $\chi : \mathbb{Z} \to \mathbb{N}$ given by $\chi(z) = z - b$. Now, use Well–Ordering Principle on $\chi(B)$.

Now, a proof similar to previous part works.

(b) The set Root2 $= \{q | q \in \mathbb{Q}, q^2 < 2\}$ is non–empty (it has -1), has an upper bound ($+2$ is an upper bound), but the set Root2 has no least upper bound. †

(c) Take $S = \mathbb{Z} \cup \{\frac{1}{2}\}$ is one such example. ‡ For extra–credit, prove that for any finite collection of positive real numbers $\Delta = \{\delta_1, \delta_2, \ldots, \delta_n\}$, the set $\Delta \cdot \mathbb{Z} = \{\sum_1^n z_i\delta_i | z_i \in \mathbb{Z}\}$ has the LUB property.

(d) Take $T = \{z | z \in \mathbb{R}, z < 0\}$. This set is non–empty (it has -1), has an upper bound ($+2$ is an upper bound), but the set T has no least upper bound in $\mathbb{R} \setminus \{0\}$. ⌝

9. For any non–zero $a \in \mathbb{R}$, show that $0 < a^2$. Also prove the corollary: $0 < n$ for every natural n.

Hint:

Case (a) If $0 < a$, i.e.. $a \in \mathbb{P}$, by axioms for *complete ordered field* (in particular Axiom 1(d) on lecture # 1), we have $a \cdot a \in \mathbb{P}$, viz., $0 < a^2$.

Case (b) If $a < 0$, then by definition $b = -a \in \mathbb{P}$. Using the fact that square of a postive real is positive (proved Case (a)), we have $0 < b^2 = (-a)\cdot(-a)$. Using Problem 4(b) of this tutorial sheet, $(-a)\cdot(-a) = a^2$. Latter two statements imply that $0 < a^2$.

By 'Law of Trichotomy' above two cases suffice.

10. Prove the Archimedean Property for reals. Explicitly, for any real number x, there exists a natural N such that $x < N$.

Hint:

See 2.4.3, Chapter 2 of Bartle and Sherbert's Real Analysis.

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11. Suppose α and β are two real numbers. In this question, we will define the sum $\alpha + \beta$.

Recall real numbers as perfect measurements, as defined in class. Let $I_1 \supset I_2 \supset I_3 \supset \cdots$ be a representative perfect measurement for the equivalence class α and $J_1 \supset J_2 \supset J_3 \supset \cdots$ be a representative for β.

(a) Can you think of a sequence of measurements $L_1 \supset L_2 \supset L_3 \supset \cdots$ corresponding to $\alpha + \beta$?

(b) Prove that your sequence of measurements is a perfect measurement.

(c) Prove that the above definition for sum is independent of the choice of representatives for α and β.

12. Let $f : \mathbb{N} \to \mathbb{N}$ be any polynomial function of degree k, i.e. k is any natural number and $f(n) = a_0 + a_1 \cdot n + a_2 \cdot n^2 + \cdots + a_k \cdot n^k$, for naturals $a_i \in \mathbb{N}$. Show that $\lim\limits_{n\to\infty} \frac{f(n)}{n!} = 0$.

13. If we did not know how to do Problem 1(c) above, we might have added this as axiom #10 to the nine algebraic axioms for reals. If we had done this, someone would have eventually found out that #10 is derivable or deducible from the other nine and hence #10 is *dependent* on the first nine axioms. Show that the axiom of commutativity of addition is dependent on the other eight axioms. Complete this exercise for the remaining eight axioms. Is there an example of two axioms which can be dropped and are derivable from other seven? More than two?
For a binary operation in general, is the property of commutativity derivable from associativity and vice–versa?

14. Let $S \subset \mathbb{R}$. We say that S is *complete* if for every non–empty bounded subset $T \subset S$ both $\sup T$ and $\inf T$ are in S. You are aware that $\mathbb{R}$ is complete. Prove that if S is a finite set, S is complete. Further, prove that $\mathbb{N}$ and $\mathbb{Z}$ are complete. Can you find other examples?

15. From your physics laboratory courses you must be aware that every experimental measurement of a length S (or time, mass, area, volume, etc.) is reported as $q \pm \epsilon$ in appropriate units. Here q and ϵ are both rational numbers and $\epsilon > 0$. This is interpreted as $q - \epsilon < S < q + \epsilon$. I claim that a particular stick A has length equal to 2m and another stick B has length equal to $\sqrt{2}$m. How will you experimentally verify these claims?

16. Does there exist a bamboo stick B such that (i) every stick of length at most 3m (including 3m) is shorter than B and (ii) every stick of length more than 3m is longer than B? If such a stick exists, devise an experiment to identify it. If no such stick exists, prove the same.

17. You might be familiar with integer and floating point arithmetic with computers. Can you build a computer which can handle arithmetic of real numbers?

18. Given a natural k, in how many different ways can you add k real numbers? Can you prove that all these methods yield the same answer? Using this, define the multi–variable sum $f : \mathbb{R}^k \to \mathbb{R}$, $f(x_1, x_2, \ldots, x_k) = x_1 + x_2 + \cdots + x_k$.

19. Prove that in $\mathbb{R}$, $1 \neq 0$.

20. Prove that if a real number $c > 0$, then $\frac{1}{c} > 0$.