1. Prove that the Principle of Mathematical Induction implies and is implied by The Well–Ordering Principle.

2. (a) For any two real numbers z and a, if $z + a = a$, then $z = 0$.
   (b) For any real number a, prove: $a \cdot 0 = 0$.
   (c) (Additive Cancellation Law) For any three real numbers a, x, y, if $a + x = a + y$, then show that $x = y$.
   (d) (Multiplicative Cancellation Law) For any three real numbers a, x, y, if $a \cdot x = a \cdot y$, and $a \neq 0$, then show that $x = y$.

3. If $a, b \in \mathbb{R}$, prove the following.

   (a) $a+b = 0 \Rightarrow b = -a$ (b) $-(-a) = a$ (c) $(-1) \cdot a = -a$ (d) $(-1) \cdot (-1) = 1$.

4. Prove that if $a, b \in \mathbb{R}$, then

   (a) $-(a + b) = (-a) + (-b)$
   (b) $(-a) \cdot (-b) = a \cdot b$
   (c) $1/(-a) = -(1/a)$, if $a \neq 0$
   (d) $-(a/b) = (-a)/b$, if $b \neq 0$

5. Read about properties of $\mathbb{R}$ from the text BaSh. Let a, b, c and d be reals.
   (a) If $a < b$ and $c < d$, prove that $a + c < b + d$.
   (b) If $0 < a < b$ and $0 < c < d$, prove that $0 < ac < bd$.

6. Give examples for

   (a) $S \subseteq \mathbb{Q}$ such that $\sup S \in \mathbb{Q}$
   (b) $T \subsetneq \mathbb{Q}$ such that $\sup T \notin \mathbb{Q}$
   (c) $U \subseteq \mathbb{R} \setminus \mathbb{Q}$ such that $\sup U \in \mathbb{R} \setminus \mathbb{Q}$
   (d) $V \subsetneq \mathbb{R} \setminus \mathbb{Q}$ such that $\sup V \notin \mathbb{R} \setminus \mathbb{Q}$.

7. If $S = \{\frac{1}{n} - \frac{1}{m} | n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$. Justify.

8. Let $S \subset \mathbb{R}$. Suppose $\phi \neq T \subset S$, such that T has an upper bound in S. If for every such T, there is a † least upper-bound in S, we say S has LUB property.
   (a) Show that $\mathbb{N}$ & $\mathbb{Z}$ have LUB property.
   (b) Show that $\mathbb{Q}$ doesn't have LUB property.
   (c) Give an example for S such that $\mathbb{Z} \subsetneq S \subsetneq \mathbb{R}$ which has LUB property. Justify.
   (d) Show that $\mathbb{R} \setminus \{0\}$ does NOT have the LUB property.

   †: The indefinite article "a" in English denotes "one". In mathematics, it means the existence of at least one such element.

9. For any non–zero $a \in \mathbb{R}$, show that $0 < a^2$. Also prove the corollary: $0 < n$ for every natural n.

10. Prove the Archimedean Property for reals. Explicitly, for any real number x, there exists a natural N such that $x < N$.

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11. Suppose α and β are two real numbers. In this question, we will define the sum $\alpha + \beta$.

    Recall real numbers as perfect measurements, as defined in class. Let $I_1 \supset I_2 \supset I_3 \supset \cdots$ be a representative perfect measurement for the equivalence class α and $J_1 \supset J_2 \supset J_3 \supset \cdots$ be a representative for β.
    (a) Can you think of a sequence of measurements $L_1 \supset L_2 \supset L_3 \supset \cdots$ corresponding to $\alpha + \beta$?
    (b) Prove that your sequence of measurements is a perfect measurement.
    (c) Prove that the above definition for sum is independent of the choice of representatives for α and β.

12. Let $f : \mathbb{N} \to \mathbb{N}$ be any polynomial function of degree k, i.e. k is any natural number and $f(n) = a_0 + a_1 \cdot n + a_2 \cdot n^2 + \cdots + a_k \cdot n^k$, for naturals $a_i \in \mathbb{N}$. Show that $\lim\limits_{n\to\infty} \frac{f(n)}{n!} = 0$.

13. If we did not know how to do Problem 1(c) above, we might have added this as axiom #10 to the nine algebraic axioms for reals. If we had done this, someone would have eventually found out that #10 is derivable or deducible from the other nine and hence #10 is *dependent* on the first nine axioms. Show that the axiom of commutativity of addition is dependent on the other eight axioms. Complete this exercise for the remaining eight axioms. Is there an example of two axioms which can be dropped and are derivable from other seven? More than two?
For a binary operation in general, is the property of commutativity derivable from associativity and vice–versa?

14. Let $S \subset \mathbb{R}$. We say that S is *complete* if for every non–empty bounded subset $T \subset S$ both $\sup T$ and $\inf T$ are in S. You are aware that $\mathbb{R}$ is complete. Prove that if S is a finite set, S is complete. Further, prove that $\mathbb{N}$ and $\mathbb{Z}$ are complete. Can you find other examples?

15. From your physics laboratory courses you must be aware that every experimental measurement of a length S (or time, mass, area, volume, etc.) is reported as $q \pm \epsilon$ in appropriate units. Here q and ϵ are both <u>rational</u> numbers and $\epsilon > 0$. This is interpreted as $q - \epsilon < S < q + \epsilon$. I claim that a particular stick A has length equal to 2m and another stick B has length equal to $\sqrt{2}$m. How will you experimentally verify these claims?

16. Does there exist a bamboo stick B such that (i) every stick of length at most 3m (including 3m) is shorter than B and (ii) every stick of length more than 3m is longer than B? If such a stick exists, devise an experiment to identify it. If no such stick exists, prove the same.

17. You might be familiar with integer and floating point arithmetic with computers. Can you build a computer which can handle arithmetic of real numbers?

18. Given a natural k, in how many different ways can you add k real numbers? Can you prove that all these methods yield the same answer? Using this, define the multi–variable sum $f : \mathbb{R}^k \to \mathbb{R}$, $f(x_1, x_2, \ldots, x_k) = x_1 + x_2 + \cdots + x_k$.

1. If α and β are reals such that $0 < \alpha < 1 < \beta$, show that $0 < \cdots < \alpha^3 < \alpha^2 < \alpha < 1 < \beta < \beta^2 < \beta^3 < \cdots$ i.e., show that $0 < \alpha^{n+1} < \alpha^n < \alpha < 1 < \beta < \beta^n < \beta^{n+1}$, for every natural $n \geq 2$. Does this automatically imply $\lim \alpha^n = 0$?

2. Applying the definition of limit of a sequence, prove $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$.

3. Complete the sentence: A sequence (x_n) in $\mathbb{R}$ does not converge to $x \in \mathbb{R}$, i.e., $x_n \not\to x$ if ...

4. Establish the convergence or divergence of the following sequences.

   (a) $a_n = 3^n$ (b) $b_n = \frac{n^2}{n+1}$ (c) $c_n = \frac{(-1)^n n}{n+1}$ (d) $d_n = (2 + \frac{1}{n})^2$

5. Determine the limits of the following sequences.

   (a) (n^{1/n^2}) (b) $((n!)^{1/n^2})$

6. Show that if X and Y are sequences such that X is convergent and Y is divergent, then $X + Y$ is divergent.

7. Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.

8. Let $x_n := \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ for each $n \in \mathbb{N}$. Prove that (x_n) converges.

9. Show that the following sequences are divergent.

   (a) $(1 - (-1)^n + 1/n)$ (b) $(\sin(n\pi/4))$

10. Show directly from the definition that the following sequences are cauchy.

    (a) $a_n = \frac{n+1}{n}$ (b) $b_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

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11. Investigate the convergence or divergence of $e_n = \frac{f(n)}{g(n)}$, f and g are polynomial functions from naturals to naturals and $g(n) \neq 0$ for every natural n.

12. If $a > 0$ and $b > 0$, show that the $\lim_{n\to\infty} (\sqrt{(n+a)(n+b)} - n) = \frac{1}{2}(a + b)$.

13. Let $X = (x_n)$ and $Y = (y_n)$ be two given sequences and let $Z = (z_n)$ be the sequence which is the *even shuffle* of X and Y. That is, $z_1 = x_1, z_2 = y_1, z_3 = x_2, z_4 = y_2, \ldots$. Show that Z is convergent if and only if both X and Y are convergent to the same limit.

14. Let $X = (x_n)$ be any sequence of reals. For each natural p, define the p–th *difference of* X denoted by $\mathfrak{D}^p X$ to be the sequence $\mathfrak{D}^p X = (\mathfrak{d}^p x_n)$ given by $\mathfrak{d}^p x_n = x_{n+p} - x_n$ for each natural n. Compare the following definitions:

    X is *goochy* of *order* p_0 if $\mathfrak{D}^{p_0} X \to 0$ for a natural p_0.

    X is *poochy* if $\mathfrak{D}^p X \to 0$ for every natural p.

    X is *versachy* if $\mathfrak{D}^p X \to 0$ uniformly in natural p, i.e. for every $\epsilon > 0$, there exists a natural N (independent of p) such that for each natural p and every natural $n > N$, we have $|x_{n+p} - x_n| < \epsilon$.

    In the business of sequences, cauchy and versachy are equivalent names. However, versachy is a stronger name than poochy while poochy is a stronger name than goochy of any order more than one. To demonstrate this, do:

    Every convergent/cauchy sequence is versachy, every versachy sequence is poochy while every poochy sequence is trivially goochy of every order. Also, every versachy sequence is cauchy.

    Every bounded monotonic sequence is cauchy and hence versachy, poochy and goochy of every order.

    There are examples of monotonic sequences which are poochy but not versachy/cauchy. And, there are examples of bounded sequences which are poochy but not versachy/cauchy.

    Every goochy sequence of order 1 is poochy and more generally, every goochy sequence of order p_0 is goochy of order dp_0 for any natural d. {Hint: Triangle inequality}

Given two naturals $q > p$, by the division algorithm $q = dp + r$ for unique whole numbers d and $r < p$. If $r \neq 0$, show that there are many examples of goochy sequences of order p which are not goochy of order q. This proves that for any given natural $p_0 > 1$, there are many examples of goochy sequences of order p_0 which are not poochy. {Hint: Periodic sequences}

15. Given a sequence $X = (x_n)$ of non–zero reals, define the p–th quotient of X to be the sequence x_{n+p}/x_n. Investigate properties of the convergence of such quotients in analogy with the previous question.

16. Recall that a sequence of better and better compatible measurements for a physical scalar quantity S (like length, mass, time, etc. in appropriate units) from any laboratory is reported as :
$$[l_1, r_1] \supsetneq [l_1, r_2] \supsetneq [l_3, r_3] \supsetneq \cdots \supsetneq [l_k, r_k]$$
Each measurement $[l_i, r_i]$ implies $l_i < S < r_i$, where $l_i, r_i \in \mathbb{Q}$ for $1 \leq i \leq k \in \mathbb{N}$. Further the *error* $\frac{1}{2} \cdot (r_i - l_i)$ in i–th measurement gets smaller with increasing i due to technological advancements. In this flux, can you give a mathematical definition for such an S as a collection of compatible measurements which <u>*cannot*</u> be bettered by any technological advancement? What is the domain for all possible values of S?

17. Define a *chain of naturals* Ω to be a collection
$$\Omega_k \subset \mathbb{N} \quad \text{for} \quad k \in \mathbb{N} \quad \text{satisfying} \quad \Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots$$
We call such a collection of subsets *core–free* if
$$\cap_k \Omega_k = \emptyset$$
Given a chain of naturals Ω and a sequence of reals (c_n), define $\liminf_\Omega c_n = \lim_{k\to\infty} \inf\{c_n | n \in \Omega_k\}$ and $\limsup_\Omega c_n = \lim_{k\to\infty} \sup\{c_n | n \in \Omega_k\}$. For any chain, these two are well–defined for bounded sequences. For any core-free chain, these two are equal if and only if the sequence is convergent. And if these two are equal for one chain, they are equal for all core-free chains.