1. Consider $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $g(x) = 1/x$. Recall the definition of derivative as the limit of a difference quotient. Using this, find the derivative of g at any $c \in \mathbb{R} \setminus \{0\}$.

2. (a) Given any real y, give a precise definition of its *cube–root* and denote it by $y^{\frac{1}{3}}$.

   (b) Prove that the function $\psi : \mathbb{R} \to \mathbb{R}$ given by $y \mapsto y^{\frac{1}{3}}$ is well–defined and continuous.

   (c) Show that ψ is not differentiable at zero.

   (d) Show that ψ is differentiable at every non–zero real.

3. Let I be an interval and $f : I \to \mathbb{R}$ differentiable (on I). Prove the following.

   (a) If $f' \equiv 0$ on I then f is a constant.

   (b) If f is a constant then $f' \equiv 0$ on I.

   (c) If $f' > 0$ on I then f is strictly increasing on I.

   (d) If $f' < 0$ on I then f is strictly decreasing on I.

   (e) If f is strictly increasing on I then $f' \geq 0$.

   (f) If f is strictly decreasing on I then $f' \leq 0$.

   (g) If f is increasing on I then $f' \geq 0$ on I.

   (h) If f is decreasing on I then $f' \leq 0$ on I.

   (i) If $f' \geq 0$ on I then f is increasing on I.

   (j) If $f' \leq 0$ on I then f is decreasing on I.

4. Can you find a formal definition for the graph of a function $f : A \to B$, for $A, B \subset \mathbb{R}$? Let $a < b$ be reals and $f : [a, b] \to \mathbb{R}$ be continuous. Assume the second derivative f'' exists on (a, b). Suppose that the graph of f intersects the line segment joining the points $(a, f(a))$ and $(b, f(b))$ at $(c, f(c))$ for some real c such that $a < c < b$. Show that there exists a real $\xi \in (a, b)$ satisfying $f''(\xi) = 0$.

5. (a) For a natural n, let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be real numbers satisfying $a_1 < a_2 < \cdots < a_n$. Find a differentiable $h : \mathbb{R} \to \mathbb{R}$ satisfying $h(a_i) = b_i$ for every $1 \leq i \leq n$. Next, find another one.

   (b) Let $a, b, p, q, \alpha, \beta$ be reals with $a \neq b$. Find a differentiable $g : \mathbb{R} \to \mathbb{R}$ satisfying $g(a) = p, g(b) = q, g'(a) = \alpha$ and $g'(b) = \beta$. Next, find another one.

6. Find examples for or prove non–existence of differentiable functions whose domains and ranges are from the collection
$$\{[0, 1], [0, 100], [0, 1), [0, 100), (0, 1], (0, 100], (0, 1), (0, 100), \mathbb{R}\}.$$

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7. Let f (usually written as $f(x)$) be a real non–zero polynomial. A *root* of f is a real c such that $f(c) = 0$. (Real Analysis does not recognize 'complex' roots). A root c of f is of *multiplicity* k, for a natural k, if $f(c) = 0, f'(c) = 0, f''(c) = 0, \ldots, f^{(k-1)}(c) = 0$ but $f^{(k)}(c) \neq 0$. (Here $f^{(0)}(c) = f(c), f^{(1)}(c) = f'(c), f^{(2)}(c) = f''(c), \ldots$)

   (a) Show that multiplicity of a root is well–defined for every non–zero polynomial and is at most the degree of that polynomial.

   (b) If for a natural k, $(x - c)^k$ divides f, then c is a root of f of multiplicity at least k.

   (c) Show that if c is a root of f of multiplicity k, then $(x - c)^k$ divides f but $(x - c)^{k+1}$ does not divide f.

   (d) Prove that a polynomial of degree n has at most n roots (even if multiplicities are included). You may christen this as the 'Real Fundamental Theorem of Algebra'.

   (e) For each natural $n \geq 2$, give an example for a polynomial of degree n which has strictly fewer roots than n.

   (f) By (d), every non–zero polynomial has finitely many roots. Let p be the number of roots discounting multiplicities. Indeed, let $\{c_i\}_{i=1}^p$ be *all* the real roots of a polynomial with multiplicities $\{m_i\}_{i=1}^p$. Show that $f = \left(\prod_{i=1}^{p} (x - c_i)^{m_i} \right) g$ for a polynomial g of even degree which has no real roots. Moreover, either for all x, $g(x) > 0$ or for all x, $g(x) < 0$.

   (g) Show that every polynomial of even degree has an even number of roots while every polynomial of odd degree has an odd number of roots, counting multiplicities.

   (h) Given any polynomial, $\mathbb{R}$ can be broken up into a collection of disjoint open intervals on each of which the polynomial is strictly monotonic.
   More explicitly, let l be the number of distinct roots of odd multiplicities of f'.

      i. If $l = 0$, show that f is strictly monotonic on $\mathbb{R}$.

ii. If $l > 0$, there exist uniquely defined reals $\{c_i\}_{i=1}^{l}$ such that $-\infty = c_0 < c_1 < c_2 < \cdots < c_l < c_{l+1} = \infty$ with the following property: f is strictly monotonic on each of the intervals (c_i, c_{i+1}), for every $0 \le i \le l$.

iii. In the second case above, the nature of monotonicity of f flips as you move from each interval to its subsequent.

8. For each natural k, show that the function $f_k : \mathbb{R} \to \mathbb{R}$ given by $f_k(x) = x^{\frac{1}{2k+1}}$ and the function $g_k : [0, \infty) \to \mathbb{R}$ given by $g_k(x) = x^{\frac{1}{2k}}$ are differentiable at every non–zero real and are not differentiable at zero.

9. Find an example for a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g'(0) > 0$ and g is not increasing on any interval (of non–zero length) containing 0.

10. Let m and n be natural numbers. Let $\{a_i\}_{i=1}^{n}$, $\{b_i\}_{i=1}^{n}$, $\{c_i\}_{i=1}^{n}$, ... be $m+1$ sets of real numbers with each set having n reals. Assume $a_1 < a_2 < \cdots < a_n$. Find an m times differentiable $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(a_i) = b_i,\ f'(a_i) = c_i,\ f''(a_i) = d_i, \ldots$ (upto m–th derivative) for every $1 \le i \le n$.

11. Let $a < b$ be reals and $f : [a, b] \to \mathbb{R}$ be continuous. For a natural $n \ge 3$, assume the n–th derivative $g = \frac{d^n f}{d\, x^n}$ exists on (a, b). Suppose the graph of f intersects the line segment joining $(a, f(a))$ and $(b, f(b))$ at each of $(c_1, f(c_1)),\ (c_2, f(c_2)),\ \ldots,\ (c_{n-1}, f(c_{n-1}))$ for some real numbers $a < c_1 < c_2 < \cdots c_{n-1} < b$. Show that there exists a real $\xi \in (a, b)$ satisfying $g(\xi) = 0$.

12. Let $f : A \to \mathbb{R}$ be a function over a domain A. Suppose $c \in A$ and f is differentiable at c and $f'(c) = 0$. Then either f is locally invertible with the inverse not differentiable at $f(c)$, or there exists sequences (a_n) and (b_n) such that for every n, $a_n \ne b_n$, $f(a_n) = f(b_n)$ and $a_n, b_n \to c$.

13. Let A be a domain and $f : A \to \mathbb{R}$ be locally invertible at a $c \in A$. Is it true that either f is differentiable at c or f^{-1} is differentiable at $f(c)$?

14. Let I be an interval and $f : I \to \mathbb{R}$ differentiable. Let $Z \subset I$ be finite. If $f' > 0$ on $I \setminus Z$ while $f' = 0$ on Z, then show that f is *strictly* increasing on I. One application is strict monotonicity of odd power functions.